

Towards NMHV amplitudes at strong coupling

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Abstract

Pentagon Operator Product Expansion provides a non-perturbative framework for analysis of scattering amplitudes in planar maximally supersymmetric gauge theory building up on their duality to null polygonal superWilson loop and integrability. In this paper, we construct a systematic expansion for the main ingredients of the formalism, i.e., pentagons, at large 't Hooft coupling as a power series in its inverse value. The calculations are tested against relations provided by the so-called Descent Equation which mixes transitions at different perturbative orders. We use leading order results to have a first glimpse into the structure of scattering amplitude at NMHV level at strong coupling.

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1 Introduction

The formulation of the Pentagon [1] Operator Product Expansion [2] for superWilson loop \mathcal{W}_N on a null polygonal contour paved a way for unravelling analytical structure of scattering super-amplitude \mathcal{A}_N of N particles, they are dual to [3, 4, 5, 6, 7, 8], at any value of 't Hooft coupling in planar maximally supersymmetric Yang-Mills theory. This convergent series is developed in terms of elementary excitations ψ of the color flux-tube propagating on the world-sheet stretched on the loop [1],

$$\mathcal{W}_N = \sum_{\psi_1, \dots, \psi_{3N-15}} \langle 0 | \hat{\mathcal{P}} | \psi_1 \rangle e^{-\tau_1 E_{\psi_1} + \sigma_1 p_{\psi_1} + i\phi_1 m_1} \langle \psi_1 | \hat{\mathcal{P}} | \psi_2 \rangle e^{-\tau_2 E_{\psi_2} + \sigma_2 p_{\psi_2} + i\phi_2 m_2} \dots \langle \psi_{3N-15} | \hat{\mathcal{P}} | 0 \rangle, \quad (1.1)$$

with $(N - 5)$ sets of three conformal cross ratios τ, σ, ϕ which encode the shape of the boundary. In addition to the dispersion relations $E_\psi = E_\psi(u)$, $p_\psi = p_\psi(u)$ (parametrized by the rapidity u) for ψ , which were known to all loops for quite some time [9], their form factor couplings to the contour $\langle 0 | \hat{\mathcal{P}} | \psi_i \rangle$ and transitions amplitudes $\langle \psi_i | \hat{\mathcal{P}} | \psi_j \rangle$ between adjacent squares in a geometric tessellation of the polygon were uncovered in a series of recent papers [10, 11, 12, 13, 14, 15, 16, 17, 18].

For a few notable exceptions [1, 12, 19, 20], recent literature was predominantly focused on perturbative analyses of scattering amplitudes at weak coupling where a plethora of data is available from different formalisms such as hexagon [21, 22] and heptagon [23, 24, 25, 26] bootstraps. The reason for this is that at each order in 't Hooft coupling there is only a very small number of flux-tube excitations which determine the amplitude in question. At strong coupling on the contrary, summation over their infinite number should be performed to reproduce the minimal area result obtained within the Thermodynamic Bethe Ansatz [3, 27, 28] as well as as to systematically decode all higher order corrections in $1/g$. At leading order in the inverse coupling, this was effectively demonstrated recently in Ref. [20] following the route outlined in Ref. [12] for MHV amplitudes. Since the dominant contribution at infinite coupling is essentially insensitive to the helicity of external particles involved in scattering, one anticipates to find a factorized overall minimal area prefactor in non-MHV amplitudes as well [29].

In this work, we initiate a systematic study of the flux-tube pentagons in the perturbative regime at strong coupling. Presently, we will not attempt however to unravel the structure of nonperturbative $e^{-\pi g}$ corrections, though these can be systematically accounted for upon a more thorough consideration. They will become important in the analysis of the transition region from strong to finite and then weak coupling. Compared to other excitations,—fermions, gluons and bound states thereof,—scalars, also known as holes, possess exponentially vanishing masses at strong coupling, potentially producing leading order contribution in the multi-collinear kinematics, i.e., $\tau_i \rightarrow \infty$. However, their effect in the amplitude is formally suppressed by inverse coupling relative to semiclassical string effects and, for this reason, we will ignore holes in the nonperturbative regime, though they were shown to provide an additive geometry-independent constant contribution to the minimal area due to their intricate infrared dynamics when resummed to all orders [19]. We further comment on their contribution to NMHV amplitudes in the Conclusions.

For the exception of the scalars, which are not presently discussed, the perturbative string regime corresponds to the one where the rapidity of excitations scales with 't Hooft constant, $u = 2g\hat{u}$ as g is sent to infinity while \hat{u} is kept fixed. For the gauge fields and bound states, the physical region of \hat{u} corresponds to the interval $(-1, 1)$, while for fermions, \hat{u} resides on the small

fermion sheet containing the point of the fermion at rest and thus varies over two semi-infinite segments $\hat{u} \in (-\infty, -1) \cup (1, \infty)$. It is for these values, the energy E and momentum p of these flux-tube excitations are of order one in g , i.e., $E, p \sim g^0$. Elsewhere, we are entering kinematics where they scale as a power of g (fractional for near-flat or integer for semiclassical regimes) yielding exponentially suppressed contribution to the Wilson loop.

Our subsequent consideration is organized as follows. In the next section, we start with small fermions and solve their parity even and odd flux-tube equations in inverse powers of the coupling. These are used then to construct direct and mirror S-matrices which enter as the main ingredients into the fermion-fermion pentagons. We continue in Sect. 3 with a similar consideration for gluons and their bound states. Due to complicated analytical structure on the physical sheet, we first pass to the half-mirror, or Goldstone, sheets and perform the strong coupling analysis there. In this manner, we find bound-state-bound-states pentagons. In Sect. 4, we use the results of the previous two sections to find mixed fermion-bound-states pentagons verifying consistency of our findings by means of the exchange relations. Another layer of consistency checks arises from consequences of the Descent Equation for superamplitudes in Sect. 5. Finally, we apply our construction to resum the entire series of gluon bound states and effective fermion-antifermion strong-coupling bound pairs for a particular component of NMHV amplitude, observing anticipated factorization of the minimal area from a helicity-dependent prefactor. Finally, we conclude. Several appendices contain compendium of integrals needed for calculations involved as well as a list of results which are too cumbersome to be quoted in the main text.

2 Small fermion transitions

Let us start our consideration with fermions. As we advertised in the Introduction, only the fermion living on the small Riemann sheet survives at strong coupling. In the theory of the flux-tube, the direct and mirror scattering matrices for small-fermion-small-(anti)fermion elementary excitations (and consequently their pentagon transitions) [12, 13]

$$\begin{aligned} S_{\text{ff}}(u, v) &= S_{\text{ff}}(u, v) \\ &= \exp \left(-2i f_{\text{ff}}^{(1)}(u, v) + 2i f_{\text{ff}}^{(2)}(u, v) \right), \end{aligned} \quad (2.1)$$

$$\begin{aligned} S_{*\text{ff}}(u, v) &= \frac{u - v + i}{u - v} S_{\text{ff}}(u, v) \\ &= \exp \left(2f_{\text{ff}}^{(3)}(u, v) - 2f_{\text{ff}}^{(4)}(u, v) \right), \end{aligned} \quad (2.2)$$

are determined by means of the dynamical phases

$$f_{\text{ff}}^{(1)}(u, v) = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \cos(vt) \tilde{\gamma}_{+,u}^{\text{f}}(2gt), \quad f_{\text{ff}}^{(2)}(u, v) = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \sin(vt) \gamma_{-,u}^{\text{f}}(2gt), \quad (2.3)$$

$$f_{\text{ff}}^{(3)}(u, v) = +\frac{1}{2} \int_0^\infty \frac{dt}{t} \sin(vt) \tilde{\gamma}_{-,u}^{\text{f}}(2gt), \quad f_{\text{ff}}^{(4)}(u, v) = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \cos(vt) \gamma_{+,u}^{\text{f}}(2gt). \quad (2.4)$$

which depend on the solutions to the flux-tube equations with sources specific to the type of excitation under consideration. Let us turn to the solution of the u -parity even and odd functions $\gamma_{-,u}^{\text{f}}$ and $\tilde{\gamma}_{-,u}^{\text{f}}$, respectively, at strong coupling.

2.1 General solution for even u -parity

An infinite set of even u -parity flux-tube equations for the small fermion [9, 13] can be cast in the form¹

$$\int_0^\infty dt \sin(vt) \left[\frac{\gamma_{+,u}^f(2gt)}{1 - e^{-t}} - \frac{\gamma_{-,u}^f(2gt)}{e^t - 1} \right] = \frac{1}{2} \int_0^\infty dt \sin(vt) \cos(ut), \quad (2.5)$$

$$\int_0^\infty dt \cos(vt) \left[\frac{\gamma_{-,u}^f(2gt)}{1 - e^{-t}} + \frac{\gamma_{+,u}^f(2gt)}{e^t - 1} \right] = 0. \quad (2.6)$$

These are valid for $|v| < 2g$, while the rapidity of the small fermion resides in the domain $|u| > 2g$. As was demonstrated in Refs. [30, 31], the above equations, or rather their analogues for the ground state of the flux tube,—the cusp anomalous dimension,—can be significantly simplified by performing a transformation to

$$\Gamma_u^f(\tau) \equiv \Gamma_{+,u}^f(\tau) + i\Gamma_{-,u}^f(\tau) = \left(1 + i \coth \frac{\tau}{4g} \right) \gamma_u^f(\tau), \quad (2.7)$$

where we introduced a complex flux-tube function

$$\gamma_u^f(\tau) \equiv \gamma_{+,u}^f(\tau) + i\gamma_{-,u}^f(\tau). \quad (2.8)$$

In this way the integrands in the left-hand side read

$$\frac{\gamma_{+,u}^f(2gt)}{1 - e^{-t}} - \frac{\gamma_{-,u}^f(2gt)}{e^t - 1} = \frac{1}{2} [\Gamma_{+,u}^f(2gt) + \Gamma_{-,u}^f(2gt)], \quad (2.9)$$

$$\frac{\gamma_{-,u}^f(2gt)}{1 - e^{-t}} + \frac{\gamma_{+,u}^f(2gt)}{e^t - 1} = \frac{1}{2} [\Gamma_{-,u}^f(2gt) - \Gamma_{+,u}^f(2gt)]. \quad (2.10)$$

By rescaling the rapidity $\hat{u} = u/(2g)$ and the integration variable $\tau = 2gt$, the equations cease to possess explicit dependence on the 't Hooft coupling. The dependence on the latter is induced however via analyticity conditions on their solutions as will be done in Sect. 2.3.

To solve the above equations it is instructive to first reduce them to a singular integral equation by means of a Fourier transformation [31]. Namely, we introduce the Fourier transforms of the functions involved

$$\varphi_u^f(p) = \int_{-\infty}^\infty \frac{d\tau}{2\pi} e^{ip\tau} \gamma_u^f(\tau), \quad \Phi_u^f(p) = \int_{-\infty}^\infty \frac{d\tau}{2\pi} e^{ip\tau} \Gamma_u^f(\tau). \quad (2.11)$$

Notice that since $\gamma_u^f(\tau)$ is an analytic function in the complex plane, it admits a convergent expansion in terms of Bessel functions, i.e., $\gamma(\tau) \sim \sum_n J_n(\tau)$, on the real axis. Thus the support of its Fourier transform is restricted to the interval $|p| < 1$, i.e.,

$$\varphi_u^f(p)|_{|p|>1} = 0, \quad (2.12)$$

while $\Phi_u^f(p)$ is nonvanishing on the entire real line. The inverse Fourier transform of the latter can be decomposed in terms of τ -even and odd functions

$$\Gamma_{+,u}^f(\tau) = \int_{-\infty}^\infty dp \cos(p\tau) \Phi_u^f(p), \quad \Gamma_{-,u}^f(\tau) = - \int_{-\infty}^\infty dp \sin(p\tau) \Phi_u^f(p), \quad (2.13)$$

¹This is achieved by means of the Jacobi-Anger summation formulas, and differentiation w.r.t. the rapidity v .

respectively, which due to the fact that the function Φ_u^f is real, $[\Phi_u^f(\tau)]^* = \Phi_u^f(\tau)$, correspond to the real and imaginary part of Γ_u^f .

Now we are in a position to derive an integral equation for the Fourier transform Φ_u^f . To this end, we replace the two linear combinations (2.9) and (2.10) by their right-hand sides in the flux-tube equations (2.5) and (2.6), respectively, and rescale the integration variable and rapidities as explained after Eq. (2.10). Next, we substitute the definitions (2.13) into the equations obtained in the previous step and evaluate the emerging τ -integrals which result in simple rational functions of rapidities. Adding up the two results, we find that Φ_u^f obeys the following equation

$$\begin{aligned} \Phi_u^f(p) + \int_{-1}^1 \frac{dk}{\pi} \Phi_u^f(k) \frac{\mathcal{P}}{k-p} \\ = - \int_{-\infty}^{\infty} dk \Phi_u^f(k) \frac{\theta(k^2-1)}{k-\hat{u}} - \frac{1}{2\pi} \left[\frac{1}{p-\hat{u}} + \frac{1}{p+\hat{u}} \right] \equiv J_u^f(p), \end{aligned} \quad (2.14)$$

where we changed the variable \hat{v} to $\hat{v} = p$ and the integral is defined by means of the Cauchy principal value \mathcal{P} . Due to the original domain of the validity of (2.5) and (2.6), this equation defines $\Phi_u^f(p)$ for $|p| < 1$ only. However, the inhomogeneity on its right-hand side involves $\Phi_u^f(p)$ outside of the interval $(-1, 1)$. This contribution can be found by noticing that due to the support region of φ^f , its Fourier transform behaves as $\gamma^f(\tau) \sim e^{|\tau|}$ for large complex τ . As a consequence, the integral (2.11) for $\Phi_u^f(p)$ can be computed by means of the Cauchy theorem (with residues emerging from the trigonometric prefactor) by closing the contour at infinity. This can be done however only provided $|p| > 1$, yielding

$$\Phi_u^f(p)|_{|p|>1} = \theta(p-1) \sum_{n \geq 1} c_u^{f,+}(n, g) e^{-4\pi n g(p-1)} + \theta(-p-1) \sum_{n \geq 1} c_u^{f,-}(n, g) e^{-4\pi n g(-p-1)}, \quad (2.15)$$

where

$$c_u^{\pm}(n, g) = \mp 4g \gamma_u^f(\pm i 4\pi g n) e^{-4\pi n g}. \quad (2.16)$$

Obviously it is nonperturbative in its origin and still involves unknown coefficients in its decomposition. They will be fixed in Sect. 2.3.

Summarizing, the solution to the flux-tube equation (2.14) can be rewritten as a sum of solutions to homogeneous and inhomogeneous equations following standard methods [32]

$$\Phi_u^f(p) = \frac{c}{p+1} \left(\frac{1+p}{1-p} \right)^{1/4} + \frac{1}{2} J_u^f(p) - \left(\frac{1+p}{1-p} \right)^{1/4} \int_{-1}^1 \frac{dk}{2\pi} \left(\frac{1-k}{1+k} \right)^{1/4} \frac{\mathcal{P}}{k-p} J_u^f(k). \quad (2.17)$$

One can easily verify a posteriori that this indeed solves (2.14) making use of integrals (A.14) and (A.15). Substituting the source J_u^f , and partitioning the denominator, we can evaluate the resulting integrals making use of Eqs. (A.14) and (A.15), such that

$$\Phi_u^f(p)|_{|p|<1} = \phi_u^f(p) - \sqrt{2} \left(\frac{1+p}{1-p} \right)^{1/4} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\theta(k^2-1)}{k-p} \left(\frac{k-1}{k+1} \right)^{1/4} \Phi_u^f(k), \quad (2.18)$$

with

$$\phi_u^f(p) = -\frac{1}{2\sqrt{2}\pi} \left(\frac{1+p}{1-p} \right)^{1/4} \left[\frac{1}{p-\hat{u}} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} + \frac{1}{p+\hat{u}} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} \right], \quad (2.19)$$

and where in the last term of (2.18) one has to substitute the expansion (2.15) and the constant c was set to zero to comply with properties of scattering phases. Fourier transforming back, we find the all-order expression for $\Gamma_u^f(\tau)$,

$$\begin{aligned}\Gamma_u^f(\tau) = \chi_u^f(\tau) + \sum_{n \geq 1} \frac{c_u^{f,-}(n, g)}{4\pi gn - i\tau} [-i\tau V_0(-i\tau)U_1^-(4\pi gn) + 4\pi gn V_1(-i\tau)U_0^-(4\pi gn)] \\ + \sum_{n \geq 1} \frac{c_u^{f,+}(n, g)}{4\pi gn + i\tau} [-i\tau V_0(-i\tau)U_1^+(4\pi gn) + 4\pi gn V_1(-i\tau)U_0^+(4\pi gn)] ,\end{aligned}\quad (2.20)$$

where

$$\chi_u^f(\tau) = -\frac{1}{4} \left(\frac{\hat{u} - 1}{\hat{u} + 1} \right)^{1/4} W(-i\tau, \hat{u}) - \frac{1}{4} \left(\frac{\hat{u} + 1}{\hat{u} - 1} \right)^{1/4} W(-i\tau, -\hat{u}). \quad (2.21)$$

The integral representations of the special functions involved are given in Appendix A. We will turn to defining the expansion coefficients after we address the u -parity odd case first in the next section.

2.2 General solution for odd u -parity

Up to minor modification, the odd u -parity case is analyzed in a similar manner. Starting with the flux-tube equations [9, 13]

$$\int_0^\infty dt \sin(vt) \left[\frac{\tilde{\gamma}_{+,u}^f(2gt)}{1 - e^{-t}} + \frac{\tilde{\gamma}_{-,u}^f(2gt)}{e^t - 1} \right] = 0, \quad (2.22)$$

$$\int_0^\infty dt \cos(vt) \left[\frac{\tilde{\gamma}_{-,u}^f(2gt)}{1 - e^{-t}} - \frac{\tilde{\gamma}_{+,u}^f(2gt)}{e^t - 1} \right] = \frac{1}{2} \int_0^\infty dt \cos(vt) \sin(ut), \quad (2.23)$$

we introduce a complex function $\tilde{\gamma}$ via the equation analogous to (2.8) that differs by a relative minus sign

$$\tilde{\gamma}_u^f(\tau) \equiv \tilde{\gamma}_{+,u}^f(\tau) - i\tilde{\gamma}_{-,u}^f(\tau), \quad (2.24)$$

and pass to a new function

$$\tilde{\Gamma}_u^f(\tau) \equiv \tilde{\Gamma}_{+,u}^f(\tau) - i\tilde{\Gamma}_{-,u}^f(\tau) = \left(1 + i \coth \frac{\tau}{4g} \right) \tilde{\gamma}_u^f(\tau), \quad (2.25)$$

such that

$$\frac{\tilde{\gamma}_{+,u}^f(2gt)}{1 - e^{-t}} + \frac{\tilde{\gamma}_{-,u}^f(2gt)}{e^t - 1} = \frac{1}{2} \left[\tilde{\Gamma}_{+,u}^f(2gt) - \tilde{\Gamma}_{-,u}^f(2gt) \right], \quad (2.26)$$

$$\frac{\tilde{\gamma}_{-,u}^f(2gt)}{1 - e^{-t}} - \frac{\tilde{\gamma}_{+,u}^f(2gt)}{e^t - 1} = \frac{1}{2} \left[\tilde{\Gamma}_{+,u}^f(2gt) + \tilde{\Gamma}_{-,u}^f(2gt) \right]. \quad (2.27)$$

Introducing the Fourier transforms identical to Eqs. (2.11), with however a sign difference for the τ -odd part,

$$\tilde{\Gamma}_{+,u}^f(\tau) = \int_{-\infty}^\infty dk \cos(k\tau) \tilde{\Phi}_u^f(k), \quad \tilde{\Gamma}_{-,u}^f(\tau) = \int_{-\infty}^\infty dk \sin(k\tau) \tilde{\Phi}_u^f(k), \quad (2.28)$$

we obtain the singular integral equation that $\tilde{\Phi}_u^f$ obeys

$$\begin{aligned} \tilde{\Phi}_u^f(p) + \int_{-1}^1 \frac{dk}{\pi} \tilde{\Phi}_u^f(k) \frac{\mathcal{P}}{k-p} \\ = - \int_{-\infty}^{\infty} dk \tilde{\Phi}_v^f(k) \frac{\theta(k^2-1)}{k-\hat{u}} - \frac{1}{2\pi} \left[\frac{1}{p-\hat{u}} - \frac{1}{p+\hat{u}} \right]. \end{aligned} \quad (2.29)$$

As before, we split the solution $\Phi_v^f(p)$ into two regions, the interior of the interval $(-1, 1)$ and its outside. The latter admits an infinite series representation

$$\tilde{\Phi}_u^f(p)|_{|p|>1} = \theta(p-1) \sum_{n \geq 1} \tilde{c}_u^{f,+}(n, g) e^{-4\pi n g(p-1)} + \theta(-p-1) \sum_{n \geq 1} \tilde{c}_u^{f,-}(n, g) e^{-4\pi n g(-p-1)}, \quad (2.30)$$

with the expansion coefficients

$$\tilde{c}_u^{f,\pm}(n, g) = \mp 4g \tilde{\gamma}_u^f(\pm i 4\pi g n) e^{-4\pi n g}, \quad (2.31)$$

which will be fixed in the next section. Making use of the explicit sources, the solution to Eq. (2.29) yields the function inside the interval $(-1, 1)$,

$$\tilde{\Phi}_u^f(p)|_{|p|<1} = \tilde{\phi}_u^f(p) - \sqrt{2} \left(\frac{1+p}{1-p} \right)^{1/4} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\theta(k^2-1)}{k-p} \left(\frac{k-1}{k+1} \right)^{1/4} \tilde{\Phi}_u^f(k), \quad (2.32)$$

with

$$\tilde{\phi}_u^f(p) = -\frac{1}{2\sqrt{2}\pi} \left(\frac{1+p}{1-p} \right)^{1/4} \left[\frac{1}{p-\hat{u}} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} - \frac{1}{p+\hat{u}} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} \right]. \quad (2.33)$$

Fourier transforming back, it immediately produces the all-order expression for $\tilde{\Gamma}_u^f(\tau)$,

$$\begin{aligned} \tilde{\Gamma}_u^f(\tau) = \tilde{\chi}_u^f(\tau) + \sum_{n \geq 1} \frac{\tilde{c}_u^{f,-}(n, g)}{4\pi g n - i\tau} [-i\tau V_0(-i\tau) U_1^-(4\pi g n) + 4\pi g n V_1(-i\tau) U_0^-(4\pi g n)] \\ + \sum_{n \geq 1} \frac{\tilde{c}_u^{f,+}(n, g)}{4\pi g n + i\tau} [-i\tau V_0(-i\tau) U_1^+(4\pi g n) + 4\pi g n V_1(-i\tau) U_0^+(4\pi g n)], \end{aligned} \quad (2.34)$$

where

$$\tilde{\chi}_u^f(\tau) = -\frac{1}{4} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} W(-i\tau, \hat{u}) + \frac{1}{4} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} W(-i\tau, -\hat{u}). \quad (2.35)$$

Now we are in a position to construct a quantization condition for the unknown coefficients $\tilde{c}_u^{f,\pm}$ as well as $c_u^{f,\pm}$ from the previous section.

2.3 Quantization conditions and their solutions

According to their definitions (2.7) and (2.25), $\Gamma_u^f(\tau)$ and $\tilde{\Gamma}_u^f(\tau)$, respectively, possess an infinite number of fixed zeroes on the imaginary axis at $\tau = 4\pi i g x_m$ due to the trigonometric multiplier present in both, i.e.,

$$\Gamma_u^f(4\pi i g x_m) = 0, \quad \tilde{\Gamma}_u^f(4\pi i g x_m) = 0, \quad (2.36)$$

with $x_m = (m - \frac{1}{4})$ where $m \in \mathbb{Z}$. They define quantization conditions for the expansion coefficients c_u^\pm . These can be cast in the explicit form

$$\begin{aligned} \frac{\chi_u^f(4\pi i g x_m)}{V_0(4\pi g x_m)} &= \sum_{n \geq 1} c_u^{f,-}(n, g) \frac{x_m U_1^-(4\pi g n) + n r(4\pi g x_m) U_0^-(4\pi g n)}{n + m} \\ &+ \sum_{n \geq 1} c_u^{f,+}(n, g) \frac{x_m U_1^+(4\pi g n) + n r(4\pi g x_m) U_0^+(4\pi g n)}{n - m}, \end{aligned} \quad (2.37)$$

where we divided both sides by V_0 and introduced the ratio

$$r(z) = \frac{V_1(z)}{V_0(z)}. \quad (2.38)$$

Similar relation holds for \tilde{c}_u^\pm , where one has to dress everything with tildes. An equation analogous to (2.37), but for the vacuum state describing the cusp anomalous dimension, was proposed and solved in Ref. [31]. Here we will adopt the strategy advocated there and expand Eq. (2.37) systematically in the inverse powers of the 't Hooft coupling.

Making use of the asymptotic expansion of the special functions for their large argument as given in Appendix A, the above quantization conditions split into two depending on the sign of x_m since the functions involved enjoy different asymptotic behavior subject to the condition $x_m \lesseqgtr 0$. Then the parity-even expansion coefficients admit the form

$$c_u^{f,\pm}(n, g) = (8\pi g n)^{\pm 1/4} \left[a_u^{f,\pm}(n) + \frac{b_u^{f,\pm}(n)}{4\pi g} + O(1/g^2) \right], \quad (2.39)$$

with explicit a and b being

$$a_u^{f,+}(n) = -\frac{2\Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \chi_0^{f,+}(u), \quad a_u^{f,-}(n) = -\frac{\Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \chi_0^{f,-}(u), \quad (2.40)$$

$$\begin{aligned} b_u^{f,+}(n) &= \frac{2\Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \left\{ \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \chi_0^{f,-}(u) \right. \\ &\quad \left. - \left[\frac{\pi}{16} - \frac{3}{8} \ln 2 \right] (\chi_0^{f,+}(u) - 8\chi_{10}^{f,+}(u)) + \frac{1}{32n} (3\chi_0^{f,+}(u) - 32\chi_{10}^{f,+}(u)) \right\}, \end{aligned} \quad (2.41)$$

$$\begin{aligned} b_u^{f,-}(n) &= -\frac{\Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \left\{ \left[-\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \chi_0^{f,+}(u) \right. \\ &\quad \left. + \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] (\chi_0^{f,-}(u) - 8\chi_{10}^{f,-}(u)) + \frac{1}{32n} (5\chi_0^{f,-}(u) - 32\chi_{10}^{f,-}(u)) \right\}, \end{aligned} \quad (2.42)$$

respectively. Here, we introduced inhomogeneities arising from the large coupling expansion of the left-hand side of the quantization condition

$$\frac{\chi_u^f(\pm 4\pi i g |x_m|)}{V_0(\pm 4\pi g |x_m|)} = \chi_0^{f,\pm}(u) + \frac{1}{4\pi g} \frac{\chi_1^{f,\pm}(u)}{x_m} + O(1/g^2), \quad (2.43)$$

with explicit order-by-order contributions being

$$\chi_0^{f,\pm}(u) = -\frac{1}{4} \left[\frac{1}{\hat{u} \pm 1} \left(\frac{\hat{u} + 1}{\hat{u} - 1} \right)^{1/4} - \frac{1}{\hat{u} \mp 1} \left(\frac{\hat{u} - 1}{\hat{u} + 1} \right)^{1/4} \right], \quad (2.44)$$

$$\chi_1^{f,+}(u) = -\frac{3}{16} \left[\frac{1}{(\hat{u}+1)^2} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} + \frac{1}{(\hat{u}-1)^2} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} \right], \quad (2.45)$$

$$\chi_1^{f,-}(u) = \frac{5}{16} \left[\frac{1}{(\hat{u}-1)^2} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} + \frac{1}{(\hat{u}+1)^2} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} \right]. \quad (2.46)$$

In complete analogy, the solutions to the parity-odd equation read

$$\tilde{c}_v^{f,\pm}(n, g) = (8\pi g n)^{\pm 1/4} \left[\tilde{a}_v^{f,\pm}(n) + \frac{\tilde{b}_v^{f,\pm}(n)}{4\pi g} + O(1/g^2) \right], \quad (2.47)$$

where the \tilde{a} and \tilde{b} coefficients are

$$\tilde{a}_u^{f,+}(n) = -\frac{2\ell \Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \tilde{\chi}_0^{f,+}(u), \quad \tilde{a}_u^{f,-}(n) = -\frac{\ell \Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \tilde{\chi}_0^{f,-}(u), \quad (2.48)$$

$$\begin{aligned} \tilde{b}_u^{f,+}(n) = & \frac{2\ell \Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \left\{ \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \tilde{\chi}_0^{f,-}(u) \right. \\ & \left. - \left[\frac{\pi}{16} - \frac{3}{8} \ln 2 \right] (\tilde{\chi}_0^{f,+}(u) - 8\tilde{\chi}_{10}^{f,+}(u)) + \frac{1}{32n} (3\tilde{\chi}_0^{f,+}(u) - 32\tilde{\chi}_{10}^{f,+}(u)) \right\}, \end{aligned} \quad (2.49)$$

$$\begin{aligned} \tilde{b}_u^{f,-}(n) = & -\frac{\ell \Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \left\{ \left[-\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \tilde{\chi}_0^{f,+}(u) \right. \\ & \left. + \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] (\tilde{\chi}_0^{f,-}(u) - 8\tilde{\chi}_{10}^{f,-}(u)) + \frac{1}{32n} (5\tilde{\chi}_0^{f,-}(u) - 32\tilde{\chi}_{10}^{f,-}(u)) \right\}, \end{aligned} \quad (2.50)$$

respectively, determined by another set of inhomogeneities arising in the left-hand side of the quantization condition

$$\frac{\tilde{\chi}_u^f(\pm 4\pi i g |x_m|)}{V_0(\pm 4\pi g |x_m|)} = \tilde{\chi}_0^{f,\pm}(u) + \frac{1}{4\pi g} \frac{\tilde{\chi}_1^{f,\pm}(u)}{x_m} + O(1/g^2), \quad (2.51)$$

with

$$\tilde{\chi}_0^{f,\pm}(u) = \frac{1}{4} \left[\frac{1}{\hat{u} \pm 1} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} + \frac{1}{\hat{u} \mp 1} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} \right], \quad (2.52)$$

$$\tilde{\chi}_1^{f,+}(u) = \frac{3}{16} \left[\frac{1}{(\hat{u}+1)^2} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} - \frac{1}{(\hat{u}-1)^2} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} \right], \quad (2.53)$$

$$\tilde{\chi}_1^{f,-}(u) = -\frac{5}{16} \left[\frac{1}{(\hat{u}-1)^2} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} - \frac{1}{(\hat{u}+1)^2} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} \right]. \quad (2.54)$$

The strong coupling expansion can be performed in a straightforward fashion to any required order. To save space we will not present subleading terms explicitly here.

2.4 Strong coupling expansion

Having determined the last unknown ingredients of the solutions, we can sum-up the infinite series in Eqs. (2.34), (2.34) and determine the inverse coupling expansion of the flux-tube functions Γ

and $\tilde{\Gamma}$. For further use, let us decompose the latter in terms of even and odd components with respect to τ . They are

$$\begin{aligned}\Gamma_{\pm,u}^f(\tau) &= \mp \frac{1}{4} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} W^\pm(\tau, \hat{u}) \mp \frac{1}{4} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} W^\pm(\tau, -\hat{u}) \\ &\mp \frac{\chi_0^{f,-}(u)}{4\pi g} \left(\frac{\pi}{8} + \frac{3}{4} \ln 2 \right) V_1^\pm(\tau) \pm \frac{\chi_0^{f,+}(u)}{4\pi g} \left(\frac{\pi}{8} - \frac{3}{4} \ln 2 \right) [V_1^\pm(\tau) \mp 4\tau V_0^\mp(\tau)] + O(1/g^2),\end{aligned}\quad (2.55)$$

and

$$\begin{aligned}\tilde{\Gamma}_{\pm,u}^f(\tau) &= -\frac{1}{4} \left(\frac{\hat{u}-1}{\hat{u}+1} \right)^{1/4} W^\pm(\tau, \hat{u}) + \frac{1}{4} \left(\frac{\hat{u}+1}{\hat{u}-1} \right)^{1/4} W^\pm(\tau, -\hat{u}) \\ &- \frac{\tilde{\chi}_0^{f,-}(u)}{4\pi g} \left(\frac{\pi}{8} + \frac{3}{4} \ln 2 \right) V_1^\pm(\tau) + \frac{\tilde{\chi}_0^{f,+}(u)}{4\pi g} \left(\frac{\pi}{8} - \frac{3}{4} \ln 2 \right) [V_1^\pm(\tau) \mp 4\tau V_0^\mp(\tau)] + O(1/g^2),\end{aligned}\quad (2.56)$$

where we introduced τ -even and -odd functions by decomposing $W(-i\tau, \hat{u})$ as $W(-i\tau, \hat{u}) = W^+(\tau, \hat{u}) - iW^-(\tau, \hat{u})$ and similarly for V_n , see Eqs. (A.12) and (A.13).

The $1/g$ expansion of the dynamical phases for the direct and mirror scattering matrices is now preformed in a straightforward fashion by trading γ 's for the linear combination of Γ 's according to the equations

$$\gamma_{\pm,u}^f(\tau) = \frac{\Gamma_{\pm,u}^f(\tau) \pm \coth \frac{\tau}{4g} \Gamma_{\mp,u}^f(\tau)}{1 + \coth^2 \frac{\tau}{4g}}, \quad \tilde{\gamma}_{\pm,u}^f(\tau) = \frac{\tilde{\Gamma}_{\pm,u}^f(\tau) \mp \coth \frac{\tau}{4g} \tilde{\Gamma}_{\mp,u}^f(\tau)}{1 + \coth^2 \frac{\tau}{4g}}, \quad (2.57)$$

and expanding the integrands of (2.3) and (2.4) for fixed τ . Substituting the above solutions into the scattering phases, we find

$$f_{\text{ff}}^{(\alpha)}(u_1, u_2) = \frac{1}{32g} \left\{ A_{\text{ff}}^{(\alpha)}(\hat{u}_1, \hat{u}_2) + \frac{1}{4g} \left[B_{\text{ff}}^{(\alpha)}(\hat{u}_1, \hat{u}_2) + \frac{3 \ln 2}{2\pi} C_{\text{ff}}^{(\alpha)}(\hat{u}_1, \hat{u}_2) \right] + O(1/g^2) \right\}, \quad (2.58)$$

($\alpha = 1, 2, 3, 4$) with explicit functions deferred to Appendix C.1 due to their length. We verified their correctness by means of the exchange relations that imply that $f_{\text{ff}}^{(2)}(u_1, u_2) = f_{\text{ff}}^{(1)}(u_2, u_1)$ as well as symmetry of the mirror phases $f_{\text{ff}}^{(3)}(u_1, u_2) = f_{\text{ff}}^{(3)}(u_2, u_1)$ and $f_{\text{ff}}^{(4)}(u_1, u_2) = f_{\text{ff}}^{(4)}(u_2, u_1)$. Further checks will be performed below.

At leading order, the fermion-fermion S-matrix and its mirror read

$$\ln S_{\text{ff}}(u_1, u_2) = -\frac{i}{8g(\hat{u}_1 - \hat{u}_2)} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - 2 \right], \quad (2.59)$$

$$\ln S_{*\text{ff}}(u_1, u_2) = \frac{1}{8g(\hat{u}_1 - \hat{u}_2)} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + 2i \right], \quad (2.60)$$

with fermion-antifermion related to them via Eqs. (2.1) and (2.2). They agree with earlier results of [33]. While the subleading terms are new.

The small-fermion–small-(anti)fermion pentagons

$$\begin{aligned} P_{\text{f|f}}(u_1|u_2) &= \frac{i(1 - \hat{x}_{\text{f}}[\hat{u}_1]\hat{x}_{\text{f}}[\hat{u}_2])}{2g(\hat{u}_1 - \hat{u}_2)} P_{\text{f|f}}(u_1|u_2) \\ &= \frac{i\sqrt{1 - \hat{x}_{\text{f}}[\hat{u}_1]\hat{x}_{\text{f}}[\hat{u}_2]}}{2g(\hat{u}_1 - \hat{u}_2)} \exp\left(-if_{\text{ff}}^{(1)}(u_1, u_2) + if_{\text{ff}}^{(2)}(u_1, u_2) - f_{\text{ff}}^{(3)}(u_1, u_2) + f_{\text{ff}}^{(4)}(u_1, u_2)\right), \end{aligned} \quad (2.61)$$

and the measure read

$$\mu_{\text{f}}(u) = -\frac{1}{\sqrt{1 - \hat{x}_{\text{f}}^2[\hat{u}]}} \exp\left(f_{\text{ff}}^{(3)}(u, u) - f_{\text{ff}}^{(4)}(u, u)\right), \quad (2.62)$$

in terms of the found phases (2.58). Here we introduced a more natural from the point of view of fermions small fermion Zhukowski variable $x_{\text{f}}[u] = \frac{1}{2}(u - \sqrt{u^2 - (2g)^2})$ rescaled with the 't Hooft coupling $x_{\text{f}}[u] = g\hat{x}_{\text{f}}[\hat{u}]$

$$\hat{x}_{\text{f}} = \hat{u} - \sqrt{\hat{u}^2 - 1}. \quad (2.63)$$

To avoid repetitious formulas, we will not display the $1/g$ expansion of pentagons explicitly which merely reduces to the substitution of Eq. (2.58) with (C.1) – (C.9) into the above formulas, however, we write down the measure to the $O(1/g^2)$ order, which requires taking a limit,

$$\mu_{\text{f}}(u) = -\frac{1}{\sqrt{1 - \hat{x}_{\text{f}}^2[\hat{u}]}} \exp\left(\frac{1}{16g} \frac{1}{\hat{u}^2 - 1} \left[1 - \frac{\pi + 12 \ln 2(\hat{u}^2 + 1)}{16\pi g(\hat{u}^2 - 1)}\right] + O(1/g^3)\right). \quad (2.64)$$

3 Gluon transitions

Now we are turning to the gauge fields and their bound states. The direct and mirror S-matrices for opposite and like helicity gluon stacks can be constructed by a fusion procedure, as was previously reported in Ref. [14]. To avoid complications in algebra due to presence of an infinite number of cuts on the physical sheet, it was instructive to pass to the Goldstone sheet [36], which is half-way between the real and mirror kinematics. The result of the analysis is summarized in the equations

$$\begin{aligned} S_{\ell_1 \bar{\ell}_2}(u_1, u_2) &= s_{\ell_1 \ell_2}^{-1}(u_1, u_2) S_{\ell_1 \ell_2}(u_1, u_2) \\ &= \exp\left(2i\sigma_{\ell_1 \ell_2}(u_1, u_2) - 2if_{\ell_1 \ell_2}^{(1)}(u_1, u_2) + 2if_{\ell_1 \ell_2}^{(2)}(u_1, u_2)\right), \end{aligned} \quad (3.1)$$

$$\begin{aligned} S_{*\ell_1 \bar{\ell}_2}(u_1, u_2) &= S_{*\ell_2 \ell_1}(u_2, u_1) \\ &= s_{*\ell_1 \bar{\ell}_2}(u_1, u_2) \exp\left(2\widehat{\sigma}_{\ell_1 \ell_2}(u_1, u_2) + 2f_{\ell_1 \ell_2}^{(3)}(u_1, u_2) - 2f_{\ell_1 \ell_2}^{(4)}(u_1, u_2)\right), \end{aligned} \quad (3.2)$$

respectively. Here the rational prefactor for the same-helicity S-matrix is

$$\begin{aligned} s_{\ell_1 \ell_2}(u_1, u_2) &= \frac{\Gamma\left(1 + \frac{\ell_1 + \ell_2}{2} - iu_1 + iu_2\right) \Gamma\left(\frac{\ell_1 + \ell_2}{2} - iu_1 + iu_2\right)}{\Gamma\left(1 + \frac{\ell_1 + \ell_2}{2} + iu_1 - iu_2\right) \Gamma\left(\frac{\ell_1 + \ell_2}{2} + iu_1 - iu_2\right)} \\ &\times \frac{\Gamma\left(1 + \frac{\ell_1 - \ell_2}{2} + iu_1 - iu_2\right) \Gamma\left(\frac{\ell_1 - \ell_2}{2} + iu_1 - iu_2\right)}{\Gamma\left(1 + \frac{\ell_1 - \ell_2}{2} - iu_1 + iu_2\right) \Gamma\left(\frac{\ell_1 - \ell_2}{2} - iu_1 + iu_2\right)} \end{aligned} \quad (3.3)$$

and corresponds to the scattering phase of spin- ℓ magnons for compact XXX spin chain. While in the mirror matrix, it takes the form

$$s_{*\ell_1\bar{\ell}_2}(u_1, u_2) = (-1)^{\ell_2} \frac{\Gamma\left(1 + \frac{\ell_1 - \ell_2}{2} - iu_1 + iu_2\right) \Gamma\left(\frac{\ell_1 + \ell_2}{2} + iu_1 - iu_2\right)}{\Gamma\left(1 + \frac{\ell_1 + \ell_2}{2} - iu_1 + iu_2\right) \Gamma\left(\frac{\ell_1 - \ell_2}{2} + iu_1 - iu_2\right)}. \quad (3.4)$$

The dynamical phases in the above equations, in a form slightly different compared to Ref. [14], read for direct

$$\sigma_{\ell_1\ell_2}(u_1, u_2) = \int_0^\infty \frac{dt}{t(e^t - 1)} \left[e^{-\ell_1 t/2} \sin(u_1 t) J_0(2gt) - e^{-\ell_2 t/2} \sin(u_2 t) J_0(2gt) - e^{-(\ell_1 + \ell_2)t/2} \sin((u_1 - u_2)t) \right], \quad (3.5)$$

$$f_{\ell_1\ell_2}^{(1)}(u_1, u_2) = \int_0^\infty \frac{dt}{t} e^{-\ell_1 t/2} \sin(u_1 t) \left[\frac{\gamma_{-,u_2}^g(2gt)}{1 - e^{-t}} + \frac{\gamma_{+,u_2}^g(2gt)}{e^t - 1} \right], \quad (3.6)$$

$$f_{\ell_1\ell_2}^{(2)}(u_1, u_2) = \int_0^\infty \frac{dt}{t} (e^{-\ell_1 t/2} \cos(u_1 t) - J_0(2gt)) \left[\frac{\tilde{\gamma}_{+,u_2}^g(2gt)}{1 - e^{-t}} + \frac{\tilde{\gamma}_{-,u_2}^g(2gt)}{e^t - 1} \right]. \quad (3.7)$$

and mirror cases

$$\hat{\sigma}_{\ell_1\ell_2}(u_1, u_2) = \int_0^\infty \frac{dt}{t(1 - e^{-t})} \left[e^{-\ell_1 t/2} \cos(u_1 t) J_0(2gt) + e^{-\ell_2 t/2} \cos(u_2 t) J_0(2gt) - e^{-(\ell_1 + \ell_2)t/2} \cos((u_1 - u_2)t) - J_0^2(2gt) \right], \quad (3.8)$$

$$f_{\ell_1\ell_2}^{(3)}(u_1, u_2) = - \int_0^\infty \frac{dt}{t} e^{-\ell_1 t/2} \sin(u_1 t) \left[\frac{\tilde{\gamma}_{-,u_2}^g(2gt)}{1 - e^{-t}} - \frac{\tilde{\gamma}_{+,u_2}^g(2gt)}{e^t - 1} \right], \quad (3.9)$$

$$f_{\ell_1\ell_2}^{(4)}(u_1, u_2) = + \int_0^\infty \frac{dt}{t} (e^{-\ell_1 t/2} \cos(u_1 t) - J_0(2gt)) \left[\frac{\gamma_{+,u_2}^g(2gt)}{1 - e^{-t}} - \frac{\gamma_{-,u_2}^g(2gt)}{e^t - 1} \right]. \quad (3.10)$$

Though it is not obvious from the above representation, the exchange relations [37, 13] imply certain symmetry properties of the dynamical phases. Namely, under the permutation of arguments (and spin labels ℓ), they change as

$$f_{\ell_1\ell_2}^{(1)}(u_1, u_2) = f_{\ell_2\ell_1}^{(2)}(u_2, u_1), \quad f_{\ell_1\ell_2}^{(3)}(u_1, u_2) = f_{\ell_2\ell_1}^{(3)}(u_2, u_1), \quad f_{\ell_1\ell_2}^{(4)}(u_1, u_2) = f_{\ell_2\ell_1}^{(4)}(u_2, u_1). \quad (3.11)$$

These will be used below as a verification of results obtained at strong coupling. The above expression are well suited to the current strong-coupling analysis, however, we have to transform them first.

3.1 Passing to Goldstone sheet

As we just mentioned above, the physical sheet in the complex u plane possesses an infinite number of cuts $[-2g, 2g]$ stacked up with the interval i along the imaginary axis. For ℓ -gluon bound state, they start from $|\Im[u]| = \ell/2$ and go up/downwards. In the strong-coupling limit, one immediately finds oneself in a predicament, since all of the cuts collapse into one on the real axis pinching the physical region of rapidities $-2g < u < 2g$. To overcome this complication one has to stay in the latter region but keep away from all of the cuts. This is possible provided

one passes to the Goldstone sheet by moving upwards through the first Zhukowski cut in the upper half-plane of u . A distinguished feature of this sheet is that it has only a finite number of cuts ranging from $-\ell/2$ to $\ell/2$. So one can safely navigate away from $[-2g + i\ell/2, 2g + i\ell/2]$ to $\Im m[u] > \ell/2$ still staying in the strip. When on the Goldstone sheet, one takes the strong coupling limit, and then one can always undo the analytic continuation afterwards and go back to the physical sheet.

According to this discussion, we perform the analytic continuation $u \xrightarrow{G} u + i\ell/2 + i0_+ \rightarrow u^G = u$ for $|u| < 2g$ and immediately find for the flux-tube equations of even

$$\int_0^\infty dt \sin(vt) \left[\frac{\gamma_{+,u}^G(2gt)}{1 - e^{-t}} - \frac{\gamma_{-,u}^G(2gt)}{e^t - 1} \right] = \frac{1}{2} \int_0^\infty dt \sin(vt) \frac{\sinh \frac{\ell t}{2}}{\sinh \frac{t}{2}} e^{iut+t/2}, \quad (3.12)$$

$$\int_0^\infty dt \cos(vt) \left[\frac{\gamma_{-,u}^G(2gt)}{1 - e^{-t}} + \frac{\gamma_{+,u}^G(2gt)}{e^t - 1} \right] = \frac{1}{2} \int_0^\infty dt \cos(vt) \frac{\sinh \frac{\ell t}{2}}{\sinh \frac{t}{2}} e^{iut-t/2}, \quad (3.13)$$

and odd parity

$$\int_0^\infty dt \sin(vt) \left[\frac{\tilde{\gamma}_{+,u}^G(2gt)}{1 - e^{-t}} + \frac{\tilde{\gamma}_{-,u}^G(2gt)}{e^t - 1} \right] = \frac{1}{2i} \int_0^\infty dt \sin(vt) \frac{\sinh \frac{\ell t}{2}}{\sinh \frac{t}{2}} e^{iut-t/2}, \quad (3.14)$$

$$\int_0^\infty dt \cos(vt) \left[\frac{\tilde{\gamma}_{-,u}^G(2gt)}{1 - e^{-t}} - \frac{\tilde{\gamma}_{+,u}^G(2gt)}{e^t - 1} \right] = \frac{1}{2i} \int_0^\infty dt \cos(vt) \frac{\sinh \frac{\ell t}{2}}{\sinh \frac{t}{2}} e^{iut+t/2}, \quad (3.15)$$

respectively, in agreement with Ref. [14]. Again we repeat that these are valid for $|v| < 2g$ and $\Im m[u] > \frac{\ell}{2}$. Notice that the sources are now complex. This will lead to minor differences in the analysis that follows.

The stack-(anti)stack S-matrix with both rapidities on the Goldstone sheet then reads

$$S_{GG}(u_1, u_2) = s_{\ell_1 \ell_2}(u_1, u_2) S_{G\bar{G}}(u_1, u_2) \quad (3.16)$$

$$= s_{\ell_1 \ell_2}(u_1, u_2) \exp \left(-2if_{GG}^{(1)}(u_1, u_2) + 2if_{GG}^{(2)}(u_1, u_2) \right),$$

$$S_{*GG}(u_1, u_2) = S_{*G\bar{G}}(u_2, u_1) \quad (3.17)$$

$$= s_{*\ell_1 \bar{\ell}_2}(u_1, u_2) \exp \left(2f_{GG}^{(3)}(u_1, u_2) - 2f_{GG}^{(4)}(u_1, u_2) \right).$$

The mirror symmetry of the flux tube allows one to establish the above relation (3.17) between the mirror matrices with opposite and like helicities, which can be easily verified from the diagrammatic representation of the latter. Though it is not transparent from the notations in the relation (3.17), we implied one has to interchange ℓ_1 and ℓ_2 as well. Here the scattering phases are

$$f_{GG}^{(1)}(u_1, u_2) = i \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell_1 t}{2} \left[\frac{\gamma_{-,u_2}^G(2gt)}{1 - e^{-t}} + \frac{\gamma_{+,u_2}^G(2gt)}{e^t - 1} \right], \quad (3.18)$$

$$f_{GG}^{(2)}(u_1, u_2) = - \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell_1 t}{2} \left[\frac{\tilde{\gamma}_{+,u_2}^G(2gt)}{1 - e^{-t}} + \frac{\tilde{\gamma}_{-,u_2}^G(2gt)}{e^t - 1} \right], \quad (3.19)$$

$$f_{GG}^{(3)}(u_1, u_2) = -i \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell_1 t}{2} \left[\frac{\tilde{\gamma}_{-,u_2}^G(2gt)}{1 - e^{-t}} - \frac{\tilde{\gamma}_{+,u_2}^G(2gt)}{e^t - 1} \right], \quad (3.20)$$

$$f_{\text{GG}}^{(4)}(u_1, u_2) = - \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell_1 t}{2} \left[\frac{\gamma_{+,u_2}^{\text{G}}(2gt)}{1 - e^{-t}} - \frac{\gamma_{-,u_2}^{\text{G}}(2gt)}{e^t - 1} \right], \quad (3.21)$$

and possess only a finite number of cuts as expected.

3.2 General solution for even u -parity

In complete analogy with the fermionic case discussed in the preceding sections, we change the basis of functions as in Eq. (2.7) and then Fourier transform their linear combination as

$$\Gamma_u^{\text{G}}(\tau) = \Gamma_{u,+}^{\text{G}}(\tau) + i\Gamma_{u,-}^{\text{G}}(\tau) = \int_{-\infty}^{\infty} dk e^{-ik\tau} \Phi_u^{\text{G}}(k). \quad (3.22)$$

Here the function $\Phi_u^{\text{G}}(k)$ is complex contrary to the analogous one for the fermion by virtue of a similar property of the sources on the Goldstone sheet. The flux-tube equation for the former is then rewritten in the form

$$\begin{aligned} \Phi_u^{\text{G}}(p) + \int_{-1}^1 \frac{dk}{\pi} \frac{\mathcal{P}}{k-p} \Phi_u^{\text{G}}(k) = - \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{\theta(k^2-1)}{k-p} \Phi_u^{\text{G}}(k) \\ - \frac{1}{2\pi} \sum_{n=0}^{\ell-1} \left[\frac{1}{p + \hat{u}^{[\ell-2n-2]} + i0} + \frac{1}{p - \hat{u}^{[\ell-2n-2]} - i0} + \frac{i}{p + \hat{u}^{[\ell-2n]} + i0} - \frac{i}{p - \hat{u}^{[\ell-2n]} - i0} \right], \end{aligned} \quad (3.23)$$

with the traditional convention used for the shifted (and rescaled) rapidity variable

$$\hat{u}^{[\pm\ell]} \equiv \hat{u} \pm \frac{i}{4g} \ell. \quad (3.24)$$

The solution for the interior region reads

$$\Phi_u^{\text{G}}(p)|_{|p|<1} = \phi_u^{\text{G}}(p) - \frac{1}{\sqrt{2}} \left(\frac{1+p}{1-p} \right)^{1/4} \int_{-\infty}^{\infty} \frac{dk}{\pi} \left(\frac{k-1}{k+1} \right)^{1/4} \frac{\theta(k^2-1)}{k-p} \Phi_u^{\text{G}}(k), \quad (3.25)$$

where²

$$\begin{aligned} \phi_u^{\text{G}}(p) = \frac{1}{2\sqrt{2}} \sum_{n=0}^{\ell-1} \left\{ e^{-i\pi/4} \delta(p - \hat{u}^{[\ell-2n-2]}) + e^{i\pi/4} \delta(p + \hat{u}^{[\ell-2n-2]}) \right. \\ \left. - e^{i\pi/4} \delta(p - \hat{u}^{\ell-2n}) - e^{-i\pi/4} \delta(p + \hat{u}^{[\ell-2n]}) \right. \\ \left. - \frac{1}{\pi} \left(\frac{1+p}{1-p} \right)^{1/4} \left[e^{-i\pi/4} \frac{\mathcal{P}}{p + \hat{u}^{[\ell-2n-2]}} \left(\frac{1 + \hat{u}^{[\ell-2n-2]}}{1 - \hat{u}^{[\ell-2n-2]}} \right)^{1/4} + e^{i\pi/4} \frac{\mathcal{P}}{p - \hat{u}^{[\ell-2n-2]}} \left(\frac{1 - \hat{u}^{[\ell-2n-2]}}{1 + \hat{u}^{[\ell-2n-2]}} \right)^{1/4} \right] \right. \\ \left. - \frac{1}{\pi} \left(\frac{1+p}{1-p} \right)^{1/4} \left[e^{-i\pi/4} \frac{\mathcal{P}}{p + \hat{u}^{[\ell-2n]}} \left(\frac{1 + \hat{u}^{[\ell-2n]}}{1 - \hat{u}^{[\ell-2n]}} \right)^{1/4} + e^{i\pi/4} \frac{\mathcal{P}}{p - \hat{u}_{n,\ell}} \left(\frac{1 - \hat{u}^{[\ell-2n]}}{1 + \hat{u}^{[\ell-2n]}} \right)^{1/4} \right] \right\}. \end{aligned} \quad (3.26)$$

²A formula for the partition of the product of principal value poles becomes handy here,

$$\frac{\mathcal{P}}{x-a} \frac{\mathcal{P}}{x-b} = \frac{\mathcal{P}}{a-b} \left[\frac{\mathcal{P}}{x-a} - \frac{\mathcal{P}}{x-b} \right] + \pi^2 \delta(a-b) \delta(x-a).$$

For the exterior domain, we have as before the series representation

$$\Phi_v^G(p)|_{|p|>1} = \theta(p-1) \sum_{n \geq 1} c_v^{G,+}(n, g) e^{-4\pi n g(p-1)} + \theta(-p-1) \sum_{n \geq 1} c_v^{G,-}(n, g) e^{-4\pi n g(-p-1)}. \quad (3.27)$$

Fourier transforming back to Γ , we deduce

$$\begin{aligned} \Gamma_u^G(\tau) &= \chi_u^G(\tau) + \sum_{n \geq 1} \frac{c_u^{G,-}(n, g)}{4\pi g n - i\tau} [-i\tau V_0(-i\tau) U_1^-(4\pi g n) + 4\pi g n V_1(-i\tau) U_0^-(4\pi g n)] \\ &\quad + \sum_{n \geq 1} \frac{c_u^{G,+}(n, g)}{4\pi g n + i\tau} [-i\tau V_0(-i\tau) U_1^+(4\pi g n) + 4\pi g n V_1(-i\tau) U_0^+(4\pi g n)], \end{aligned} \quad (3.28)$$

with

$$\begin{aligned} \chi_u^G(\tau) &= \int_{-1}^1 dk e^{-i\tau k} \phi_u^G(k) = \frac{1}{4} \sum_{n=0}^{\ell-1} \left\{ 2\sqrt{2} \cos\left(\tau \hat{u}^{[\ell-2n-2]} + \frac{\pi}{4}\right) - 2\sqrt{2} \cos\left(\tau \hat{u}^{[\ell-2n]} - \frac{\pi}{4}\right) \right. \\ &\quad - e^{-i\pi/4} \left(\frac{1 + \hat{u}^{[\ell-2n-2]}}{1 - \hat{u}^{[\ell-2n-2]}}\right)^{1/4} W(-i\tau, -\hat{u}^{[\ell-2n-2]}) - e^{i\pi/4} \left(\frac{1 - \hat{u}^{[\ell-2n-2]}}{1 + \hat{u}^{[\ell-2n-2]}}\right)^{1/4} W(-i\tau, \hat{u}^{[\ell-2n-2]}) \\ &\quad \left. - e^{i\pi/4} \left(\frac{1 + \hat{u}^{[\ell-2n]}}{1 - \hat{u}^{[\ell-2n]}}\right)^{1/4} W(-i\tau, -\hat{u}^{[\ell-2n]}) - e^{-i\pi/4} \left(\frac{1 - \hat{u}^{[\ell-2n]}}{1 + \hat{u}^{[\ell-2n]}}\right)^{1/4} W(-i\tau, \hat{u}^{[\ell-2n]}) \right\}. \end{aligned}$$

3.3 General solution for odd u -parity

The flux-tube equations for the Fourier transform of $\tilde{\Gamma}_u^G(\tau)$,

$$\tilde{\Gamma}_u^G(\tau) = \tilde{\Gamma}_{u,+}^G(\tau) + i\tilde{\Gamma}_{u,-}^G(\tau) = \int_{-\infty}^{\infty} dk e^{-ik\tau} \tilde{\Phi}_u^G(k). \quad (3.29)$$

is again put in the form of a singular integral equation

$$\begin{aligned} \tilde{\Phi}_u^G(p) + \int_{-1}^1 \frac{dk}{\pi} \frac{\mathcal{P}}{k-p} \tilde{\Phi}_u^G(k) &= - \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{\theta(k^2-1)}{k-p} \tilde{\Phi}_u^G(k) \\ &\quad + \frac{1}{2\pi} \sum_{n=0}^{\ell-1} \left[\frac{1}{p + \hat{u}^{[\ell-2n-2]} + i0} - \frac{1}{p - \hat{u}^{[\ell-2n-2]} - i0} + \frac{i}{p + \hat{u}^{\ell-2n} + i0} + \frac{i}{p - \hat{u}^{\ell-2n} - i0} \right], \end{aligned} \quad (3.30)$$

whose solution is

$$\tilde{\Phi}_u^G(p)|_{|p|<1} = \tilde{\phi}_u^G(p) - \frac{1}{\sqrt{2}} \left(\frac{1+p}{1-p}\right)^{1/4} \int_{-\infty}^{\infty} \frac{dk}{\pi} \left(\frac{k-1}{k+1}\right)^{1/4} \frac{\theta(k^2-1)}{k-p} \tilde{\Phi}_u^G(k), \quad (3.31)$$

where

$$\begin{aligned} \tilde{\phi}_u^G(p) &= \frac{1}{2\sqrt{2}} \sum_{n=0}^{\ell-1} \left\{ e^{-i\pi/4} \delta(p - \hat{u}^{[\ell-2n-2]}) - e^{i\pi/4} \delta(p + \hat{u}^{[\ell-2n-2]}) \right. \\ &\quad \left. - e^{i\pi/4} \delta(p - \hat{u}^{[\ell-2n]}) + e^{-i\pi/4} \delta(p + \hat{u}^{[\ell-2n]}) \right\} \end{aligned} \quad (3.32)$$

$$\begin{aligned}
& + \frac{1}{\pi} \left(\frac{1+p}{1-p} \right)^{1/4} \left[e^{-i\pi/4} \frac{\mathcal{P}}{p + \hat{u}^{[\ell-2n-2]}} \left(\frac{1 + \hat{u}^{[\ell-2n-2]}}{1 - \hat{u}^{[\ell-2n-2]}} \right)^{1/4} - e^{i\pi/4} \frac{\mathcal{P}}{p - \hat{u}^{[\ell-2n-2]}} \left(\frac{1 - \hat{u}^{[\ell-2n-2]}}{1 + \hat{u}^{[\ell-2n-2]}} \right)^{1/4} \right] \\
& + \frac{1}{\pi} \left(\frac{1+p}{1-p} \right)^{1/4} \left[e^{i\pi/4} \frac{\mathcal{P}}{p + \hat{u}^{[\ell-2n]}} \left(\frac{1 + \hat{u}^{[\ell-2n]}}{1 - \hat{u}^{[\ell-2n]}} \right)^{1/4} - e^{-i\pi/4} \frac{\mathcal{P}}{p - \hat{u}^{[\ell-2n]}} \left(\frac{1 - \hat{u}^{[\ell-2n]}}{1 + \hat{u}^{[\ell-2n]}} \right)^{1/4} \right] \Bigg\},
\end{aligned}$$

and the outside function is again determined by the series (3.27), where one obviously dresses all coefficients with tildes. Fourier transforming it back (3.29), we deduce

$$\begin{aligned}
\tilde{\Gamma}_u^G(\tau) &= \tilde{\chi}_u^G(\tau) + \sum_{n \geq 1} \frac{\tilde{c}_u^{G,-}(n, g)}{4\pi g n - i\tau} [-i\tau V_0(-i\tau) U_1^-(4\pi g n) + 4\pi g n V_1(-i\tau) U_0^-(4\pi g n)] \\
&+ \sum_{n \geq 1} \frac{\tilde{c}_u^{G,+}(n, g)}{4\pi g n + i\tau} [-i\tau V_0(-i\tau) U_1^+(4\pi g n) + 4\pi g n V_1(-i\tau) U_0^+(4\pi g n)] , \quad (3.33)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\chi}_u^G(\tau) &= \int_{-1}^1 dk e^{-i\tau k} \tilde{\phi}_u^G(k) = \frac{1}{4} \sum_{n=0}^{\ell-1} \left\{ -i2\sqrt{2} \sin\left(\tau \hat{u}^{[\ell-2n-2]} + \frac{\pi}{4}\right) + i2\sqrt{2} \sin\left(\tau \hat{u}^{[\ell-2n]} - \frac{\pi}{4}\right) \right. \\
&+ e^{-i\pi/4} \left(\frac{1 + \hat{u}^{[\ell-2n-2]}}{1 - \hat{u}^{[\ell-2n-2]}} \right)^{1/4} W(-i\tau, -\hat{u}^{[\ell-2n-2]}) - e^{i\pi/4} \left(\frac{1 - \hat{u}^{[\ell-2n-2]}}{1 + \hat{u}^{[\ell-2n-2]}} \right)^{1/4} W(-i\tau, \hat{u}^{[\ell-2n-2]}) \\
&\left. + e^{i\pi/4} \left(\frac{1 + \hat{u}^{[\ell-2n]}}{1 - \hat{u}^{[\ell-2n]}} \right)^{1/4} W(-i\tau, -\hat{u}^{[\ell-2n]}) - e^{-i\pi/4} \left(\frac{1 - \hat{u}^{[\ell-2n]}}{1 + \hat{u}^{[\ell-2n]}} \right)^{1/4} W(-i\tau, \hat{u}^{[\ell-2n]}) \right\}.
\end{aligned}$$

3.4 Quantization conditions and their solutions

The quantization condition for the even u -parity function

$$\Gamma_u^G(4\pi i g x_m) = 0, \quad (3.34)$$

can be solved order-by-order in the inverse 't Hooft coupling with the result

$$c_u^{G,\pm}(n, g) = (8\pi g n)^{\pm 1/4} \left[a_u^{G,\pm}(n) + \frac{b_u^{G,\pm}(n)}{4\pi g} + O(1/g^2) \right], \quad (3.35)$$

where the explicit a and b coefficients are found to be

$$a_u^{G,+}(n) = -\frac{2\ell \Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \chi_0^{G,+}(u), \quad a_u^{G,-}(n) = -\frac{\ell \Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \chi_0^{G,-}(u), \quad (3.36)$$

$$\begin{aligned}
b_u^{G,+}(n) &= \frac{2\ell \Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \left\{ \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \chi_0^{G,-}(u) \right. \\
&\left. - \left[\frac{\pi}{16} - \frac{3}{8} \ln 2 \right] (\chi_0^{G,+}(u) - 8\chi_{10}^{G,+}(u)) - \chi_{11}^{G,+}(u) + \frac{1}{32n} (3\chi_0^{G,+}(u) - 32\chi_{10}^{G,+}(u)) \right\}, \quad (3.37)
\end{aligned}$$

$$b_u^{G,-}(n) = -\frac{\ell \Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \left\{ \left[-\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \chi_0^{G,+}(u) \right. \quad (3.38)$$

$$+ \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] (\chi_0^{G,-}(u) - 8\chi_{10}^{G,-}(u)) + \chi_{11}^{G,-}(u) + \frac{1}{32n} \left(5\chi_0^{G,-}(u) - 32\chi_{10}^{G,-}(u) \right) \Big\} ,$$

respectively. Here, we introduced inhomogeneities arising in the left-hand side of the quantization condition,

$$\frac{\chi_u^G(\pm 4\pi i g |x_m|)}{V_0(\pm 4\pi g |x_m|)} = \ell \chi_0^{G,\pm}(u) + \frac{\ell}{4\pi g} \left[\frac{\chi_{10}^{G,\pm}(u)}{x_m} + \chi_{11}^{G,\pm}(u) \right] + O(1/g^2) , \quad (3.39)$$

with

$$\chi_0^{G,\pm}(u) = \mp \frac{1}{2\sqrt{2}} \left[\frac{1}{1 \pm \hat{u}} \left(\frac{1+\hat{u}}{1-\hat{u}} \right)^{1/4} + \frac{1}{1 \mp \hat{u}} \left(\frac{1-\hat{u}}{1+\hat{u}} \right)^{1/4} \right] , \quad (3.40)$$

$$\chi_{11}^{G,\pm}(u) = \pm \frac{\pi}{2\sqrt{2}} \partial_{\hat{u}} \left[\frac{1}{1 \pm \hat{u}} \left(\frac{1+\hat{u}}{1-\hat{u}} \right)^{1/4} - \frac{1}{1 \mp \hat{u}} \left(\frac{1-\hat{u}}{1+\hat{u}} \right)^{1/4} \right] , \quad (3.41)$$

$$\chi_{10}^{G,+}(u) = -\frac{3}{8\sqrt{2}} \left[\frac{1}{(1+\hat{u})^2} \left(\frac{1+\hat{u}}{1-\hat{u}} \right)^{1/4} + \frac{1}{(1-\hat{u})^2} \left(\frac{1-\hat{u}}{1+\hat{u}} \right)^{1/4} \right] , \quad (3.42)$$

$$\chi_{10}^{G,-}(u) = +\frac{5}{8\sqrt{2}} \left[\frac{1}{(1-\hat{u})^2} \left(\frac{1+\hat{u}}{1-\hat{u}} \right)^{1/4} + \frac{1}{(1+\hat{u})^2} \left(\frac{1-\hat{u}}{1+\hat{u}} \right)^{1/4} \right] . \quad (3.43)$$

The defining condition for the odd u -parity coefficients

$$\tilde{\Gamma}_u^G(4\pi i g x_m) = 0 \quad (3.44)$$

provides the result

$$\tilde{c}_u^{G,\pm}(n, g) = (8\pi g n)^{\pm 1/4} \left[\tilde{a}_u^{G,\pm}(n) + \frac{\tilde{b}_u^{G,\pm}(n)}{4\pi g} + O(1/g^2) \right] , \quad (3.45)$$

with \tilde{a} and \tilde{b} being

$$\tilde{a}_u^{G,+}(n) = -\frac{2\ell \Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \tilde{\chi}_0^{G,+}(u) , \quad \tilde{a}_u^{G,-}(n) = -\frac{\ell \Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \tilde{\chi}_0^{G,-}(u) , \quad (3.46)$$

$$\begin{aligned} \tilde{b}_u^{G,+}(n) = & \frac{2\ell \Gamma(n + \frac{1}{4})}{\Gamma(n+1)\Gamma^2(\frac{1}{4})} \left\{ \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \tilde{\chi}_0^{G,-}(u) \right. \\ & \left. - \left[\frac{\pi}{16} - \frac{3}{8} \ln 2 \right] (\tilde{\chi}_0^{G,+}(u) - 8\tilde{\chi}_{10}^{G,+}(u)) - \tilde{\chi}_{11}^{G,+}(u) + \frac{1}{32n} (3\tilde{\chi}_0^{G,+}(u) - 32\tilde{\chi}_{10}^{G,+}(u)) \right\} , \end{aligned} \quad (3.47)$$

$$\begin{aligned} \tilde{b}_u^{G,-}(n) = & -\frac{\ell \Gamma(n + \frac{3}{4})}{2\Gamma(n+1)\Gamma^2(\frac{3}{4})} \left\{ \left[-\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] \tilde{\chi}_0^{G,+}(u) \right. \\ & \left. + \left[\frac{\pi}{16} + \frac{3}{8} \ln 2 \right] (\tilde{\chi}_0^{G,-}(u) - 8\tilde{\chi}_{10}^{G,-}(u)) + \tilde{\chi}_{11}^{G,-}(u) + \frac{1}{32n} (5\tilde{\chi}_0^{G,-}(u) - 32\tilde{\chi}_{10}^{G,-}(u)) \right\} , \end{aligned} \quad (3.48)$$

where the functions $\tilde{\chi}(u)$ arise from the expansion of the source $\tilde{\chi}$

$$\frac{\tilde{\chi}_u^G(\pm 4\pi i g |x_m|)}{V_0(\pm 4\pi g |x_m|)} = \ell \tilde{\chi}_0^{G,\pm}(u) + \frac{\ell}{4\pi g} \left[\frac{\tilde{\chi}_{10}^{G,\pm}(u)}{x_m} + \tilde{\chi}_{11}^{G,\pm}(u) \right] + O(1/g^2) , \quad (3.49)$$

with

$$\tilde{\chi}_0^{G,\pm}(u) = \pm \frac{1}{2\sqrt{2}} \left[\frac{1}{1 \pm \hat{u}} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} - \frac{1}{1 \mp \hat{u}} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} \right], \quad (3.50)$$

$$\tilde{\chi}_{11}^{G,\pm}(u) = \mp \frac{\pi}{2\sqrt{2}} \partial_{\hat{u}} \left[\frac{1}{1 \pm \hat{u}} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} + \frac{1}{1 \mp \hat{u}} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} \right], \quad (3.51)$$

$$\tilde{\chi}_{10}^{G,+}(u) = + \frac{3}{8\sqrt{2}} \left[\frac{1}{(1 + \hat{u})^2} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} - \frac{1}{(1 - \hat{u})^2} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} \right], \quad (3.52)$$

$$\tilde{\chi}_{10}^{G,-}(u) = - \frac{5}{8\sqrt{2}} \left[\frac{1}{(1 - \hat{u})^2} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} - \frac{1}{(1 + \hat{u})^2} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} \right]. \quad (3.53)$$

3.5 Strong coupling expansion

Using the just determined expansion coefficients, we can deduce the $1/g$ expansion of the flux-tube functions, which are

$$\begin{aligned} \Gamma_{\pm,u}^G(\tau) &= \mp \frac{\ell}{2\sqrt{2}} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} W^{\pm}(\tau, \hat{u}) \mp \frac{\ell}{2\sqrt{2}} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} W^{\pm}(\tau, -\hat{u}) + i\ell\delta_{\pm,-} \sin(\tau\hat{u}) \\ &\quad \mp \frac{\ell\pi}{4\pi g} \partial_{\hat{u}} \left[\frac{1}{2\sqrt{2}} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} W^{\pm}(\tau, \hat{u}) - \frac{1}{2\sqrt{2}} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} W^{\pm}(\tau, -\hat{u}) + i\delta_{\pm,+} \cos(\tau\hat{u}) \right] \\ &\quad \mp \frac{\ell\tilde{\chi}_0^{G,-}(u)}{4\pi g} \left(\frac{\pi}{8} + \frac{3}{4} \ln 2 \right) V_1^{\pm}(\tau) \pm \frac{\ell\tilde{\chi}_0^{G,+}(u)}{4\pi g} \left(\frac{\pi}{8} - \frac{3}{4} \ln 2 \right) [V_1^{\pm}(\tau) \mp 4\tau V_0^{\mp}(\tau)] + O(1/g^2), \end{aligned} \quad (3.54)$$

and

$$\begin{aligned} \tilde{\Gamma}_{\pm,u}^G(\tau) &= - \frac{\ell}{2\sqrt{2}} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} W^{\pm}(\tau, \hat{u}) + \frac{\ell}{2\sqrt{2}} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} W^{\pm}(\tau, -\hat{u}) - i\ell\delta_{\pm,+} \cos(\tau\hat{u}) \\ &\quad - \frac{\ell\pi}{4\pi g} \partial_{\hat{u}} \left[\frac{1}{2\sqrt{2}} \left(\frac{1 - \hat{u}}{1 + \hat{u}} \right)^{1/4} W^{\pm}(\tau, \hat{u}) + \frac{1}{2\sqrt{2}} \left(\frac{1 + \hat{u}}{1 - \hat{u}} \right)^{1/4} W^{\pm}(\tau, -\hat{u}) + i\delta_{\pm,-} \sin(\tau\hat{u}) \right] \\ &\quad - \frac{\ell\tilde{\chi}_0^{G,-}(u)}{4\pi g} \left(\frac{\pi}{8} + \frac{3}{4} \ln 2 \right) V_1^{\pm}(\tau) + \frac{\ell\tilde{\chi}_0^{G,+}(u)}{4\pi g} \left(\frac{\pi}{8} - \frac{3}{4} \ln 2 \right) [V_1^{\pm}(\tau) \mp 4\tau V_0^{\mp}(\tau)] + O(1/g^2), \end{aligned} \quad (3.55)$$

for the even and odd u -parity, respectively. Here $\delta_{++} = \delta_{--} = 1$ and $\delta_{+-} = \delta_{-+} = 0$. Substituting these solutions into the scattering phases, we find

$$f_{GG}^{(\alpha)}(u_1, u_2) = \frac{\ell_1 \ell_2}{16g} \left\{ A_{GG}^{(\alpha)}(u_1, u_2) + \frac{1}{4g} \left[B_{GG}^{(\alpha)}(u_1, u_2) + \frac{3 \ln 2}{2\pi} C_{GG}^{(\alpha)}(u_1, u_2) \right] + O(1/g^2) \right\}, \quad (3.56)$$

where the linear dependence on ℓ 's holds only up to the order in $1/g$ displayed and it becomes nonlinear beyond it. The explicit expressions are deferred to Appendix C.2.

At leading order, i.e., keeping just A 's, we find the known expressions [1, 33, 34]

$$\ln S_{GG}(u_1, u_2) = \frac{i\ell_1 \ell_2}{4g(\hat{u}_1 - \hat{u}_2)} \left[- \left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} + 2 \right], \quad (3.57)$$

$$\ln S_{*GG}(u_1, u_2) = \frac{\ell_1 \ell_2}{4g(\hat{u}_1 - \hat{u}_2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} - 2i \right], \quad (3.58)$$

for the direct and mirror S-matrices, respectively. The subleading corrections were recently verified by a direct calculation in string perturbation theory³ [35].

The pentagon transitions at strong coupling are found by substituting the above result (3.56) into the expressions for pentagons derived in Appendix B.1,

$$P_{G|G}(u_1|u_2) = w_{GG}(u_1, u_2) P_{G|\bar{G}}(u_1|u_2) \quad (3.59)$$

$$= \frac{e^{-if_{GG}^{(1)}(u_1, u_2) + if_{GG}^{(2)}(u_1, u_2) - f_{GG}^{(3)}(u_1, u_2) + f_{GG}^{(4)}(u_1, u_2)}}{s_{*\ell_1 \bar{\ell}_2}(u_1, u_2)} \left[\frac{\left(1 - \frac{1}{\hat{x}^{[-\ell_1][\hat{u}_1]\hat{x}^{[-\ell_2][\hat{u}_2]}}\right) \left(1 - \frac{1}{\hat{x}^{[\ell_1][\hat{u}_1]\hat{x}^{[\ell_2][\hat{u}_2]}}\right)}{\left(1 - \frac{1}{\hat{x}^{[\ell_1][\hat{u}_1]\hat{x}^{[-\ell_2][\hat{u}_2]}}\right) \left(1 - \frac{1}{\hat{x}^{[-\ell_1][\hat{u}_1]\hat{x}^{[\ell_2][\hat{u}_2]}}\right)} \right]^{1/2},$$

with w_{GG} given by Eq. (B.6). Here we employed the following conventions for the shifted rapidities and rescaled Zhukowski gluon variable $x[u] = g \hat{x}[\hat{u}]$,

$$\hat{x}^{[\pm \ell]}[\hat{u}] \equiv \hat{x}[\hat{u} \pm i \frac{\ell}{4g}]. \quad (3.60)$$

Finally, let us quote the bound state measure to order $O(1/g^2)$

$$\mu_G(u) = \frac{1}{\ell^2} \exp \left(-\frac{\ell^2}{8g(1 - \hat{u}^2)} \left[1 + \frac{3\pi + 12 \ln 2 (1 + \hat{u}^2)}{16\pi g(1 - \hat{u}^2)} \right] + O(1/g^3) \right). \quad (3.61)$$

Let us point out, however, that in the derivation of this expression it is important to realize that the $g \rightarrow \infty$ and the square limit, used to obtain the measure from the pentagon, do not commute. Strong coupling comes first. The above $1/\ell^2$ arises solely from the $1/s_{*\ell_1 \bar{\ell}_2}(u_1, u_2)$ prefactor in Eq. (3.59).

4 Fermion–gauge bound state transitions

The gauge bound state-(anti)fermion S-matrices are easily constructed along the same lines as the ones for a single gauge excitation and read

$$S_{\ell f}(u_1, u_2) = \frac{u_1 - u_2 - i \frac{\ell}{2}}{u_1 - u_2 + i \frac{\ell}{2}} S_{\ell \bar{f}}(u_1, u_2) \quad (4.1)$$

$$= \exp \left(-2i f_{\ell f}^{(1)}(u_1, u_2) + 2i f_{\ell f}^{(2)}(u_1, u_2) \right),$$

$$S_{* \ell f}(u_1, u_2) = \frac{u_1 - u_2 - i \frac{\ell}{2}}{u_1 - u_2 + i \frac{\ell}{2}} S_{* \ell \bar{f}}(u_1, u_2) \quad (4.2)$$

$$= \frac{g^2 (-1)^\ell}{x_f[u_2](u_1 - u_2 + i \frac{\ell}{2})} \exp \left(2f_{\ell f}^{(3)}(u_1, u_2) - 2f_{\ell f}^{(4)}(u_1, u_2) \right).$$

³We would like to thank Lorenzo Bianchi for bringing these results to our attention.

Since the prefactor, as a function of the gauge rapidity, is rational, we do not even need to pass to the Goldstone sheet to fuse this rational factor for the gluon-antifermion S-matrix. While the dynamical phases can be easily generalized for any ℓ

$$f_{\ell f}^{(1)}(u_1, u_2) = \int_0^\infty \frac{dt}{t} e^{-\ell t/2} \sin(u_1 t) \left[\frac{\gamma_{-,u_2}^f(2gt)}{1 - e^{-t}} + \frac{\gamma_{+,u_2}^f(2gt)}{e^t - 1} \right], \quad (4.3)$$

$$f_{\ell f}^{(2)}(u_1, u_2) = \int_0^\infty \frac{dt}{t} (e^{-\ell t/2} \cos(u_1 t) - J_0(2gt)) \left[\frac{\tilde{\gamma}_{+,u_2}^f(2gt)}{1 - e^{-t}} + \frac{\tilde{\gamma}_{-,u_2}^f(2gt)}{e^t - 1} \right], \quad (4.4)$$

$$f_{\ell f}^{(3)}(u_1, u_2) = - \int_0^\infty \frac{dt}{t} e^{-\ell t/2} \sin(u_1 t) \left[\frac{\tilde{\gamma}_{-,u_2}^f(2gt)}{1 - e^{-t}} - \frac{\tilde{\gamma}_{+,u_2}^f(2gt)}{e^t - 1} \right], \quad (4.5)$$

$$f_{\ell f}^{(4)}(u_1, u_2) = \int_0^\infty \frac{dt}{t} (e^{-\ell t/2} \cos(u_1 t) - J_0(2gt)) \left[\frac{\gamma_{+,u_2}^f(2gt)}{1 - e^{-t}} - \frac{\gamma_{-,u_2}^f(2gt)}{e^t - 1} \right], \quad (4.6)$$

after using exchange relations for the phases given in terms of gauge bound state flux-tube functions, see Eqs. (A.33), (A.34), (A.40) and (A.41) of Ref. [15].

4.1 Passing to Goldstone sheet

Let us pass to the Goldstone sheet since this is where we will perform the strong coupling expansion. The scattering matrices read⁴

$$\begin{aligned} S_{\text{Gf}}(u_1, u_2) &= \frac{u_1 - u_2 - i\frac{\ell}{2}}{u_1 - u_2 + i\frac{\ell}{2}} S_{\text{G}\bar{f}}(u_1, u_2) \\ &= \exp \left(-2i f_{\text{Gf}}^{(1)}(u_1, u_2) + 2i f_{\text{Gf}}^{(2)}(u_1, u_2) \right), \end{aligned} \quad (4.7)$$

$$\begin{aligned} S_{*\text{Gf}}(u_1, u_2) &= \frac{u_1 - u_2 - i\frac{\ell}{2}}{u_1 - u_2 + i\frac{\ell}{2}} S_{*\text{G}\bar{f}}(u_1, u_2) \\ &= -\frac{u_1 - u_2 - i\frac{\ell}{2}}{u_1 - u_2 + i\frac{\ell}{2}} \exp \left(2f_{\text{Gf}}^{(3)}(u_1, u_2) - 2f_{\text{Gf}}^{(4)}(u_1, u_2) \right), \end{aligned} \quad (4.8)$$

with corresponding dynamical phases being

$$\begin{aligned} f_{\text{Gf}}^{(1)}(u_1, u_2) &= i \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell t}{2} \left[\frac{\gamma_{-,u_2}^f(2gt)}{1 - e^{-t}} + \frac{\gamma_{+,u_2}^f(2gt)}{e^t - 1} \right] \\ &= -\frac{1}{2} \int_0^\infty \frac{dt}{t} \cos(u_2 t) \tilde{\gamma}_{+,u_1}^G(2gt), \\ f_{\text{Gf}}^{(2)}(u_1, u_2) &= - \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell t}{2} \left[\frac{\tilde{\gamma}_{+,u_2}^f(2gt)}{1 - e^{-t}} + \frac{\tilde{\gamma}_{-,u_2}^f(2gt)}{e^t - 1} \right] \end{aligned} \quad (4.9)$$

⁴To derive the last line in Eq. (4.8) the following formula is useful

$$\int_0^\infty \frac{dt}{t} \left[\sin(u_1^{[-\ell]} t) \sin(u_2 t) + \cos(u_1^{[-\ell]} t) \cos(u_2 t) - \cos(u_2 t) J_0(2gt) \right] = \ln \frac{g^2}{x_f[u_2](u_1^{[-\ell]} - u_2)}.$$

$$= -\frac{1}{2} \int_0^\infty \frac{dt}{t} \sin(u_2 t) \gamma_{-,u_1}^G(2gt), \quad (4.10)$$

$$\begin{aligned} f_{\text{Gf}}^{(3)}(u_1, u_2) &= -i \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell t}{2} \left[\frac{\tilde{\gamma}_{-,u_2}^f(2gt)}{1 - e^{-t}} - \frac{\tilde{\gamma}_{+,u_2}^f(2gt)}{e^t - 1} \right] \\ &= \frac{1}{2} \int_0^\infty \frac{dt}{t} \sin(u_2 t) \tilde{\gamma}_{-,u_1}^G(2gt), \end{aligned} \quad (4.11)$$

$$\begin{aligned} f_{\text{Gf}}^{(4)}(u_1, u_2) &= - \int_0^\infty \frac{dt}{t} e^{iu_1 t} \sinh \frac{\ell t}{2} \left[\frac{\gamma_{+,u_2}^f(2gt)}{1 - e^{-t}} - \frac{\gamma_{-,u_2}^f(2gt)}{e^t - 1} \right] \\ &= -\frac{1}{2} \int_0^\infty \frac{dt}{t} \cos(u_2 t) \gamma_{+,u_1}^G(2gt). \end{aligned} \quad (4.12)$$

Everywhere above it is implied that $\Im[u_1] > \ell/2$. We also showed results in terms of the gauge bound state flux-tube functions. Both of these expressions will be used in the next section along with the strong expansion constructed earlier to verify their consistency.

4.2 Strong coupling expansion

Employing the strong-coupling expansion of flux-tube functions worked out earlier, we can calculate the dynamical phases of the gluon-fermion pentagons. They admit the following form

$$f_{\text{Gf}}^{(\alpha)}(u_1, u_2) = \frac{\ell}{16\sqrt{2}g} \left\{ A_{\text{Gf}}^{(\alpha)}(\hat{u}_1, \hat{u}_2) + \frac{1}{4g} \left[B_{\text{Gf}}^{(\alpha)}(\hat{u}_1, \hat{u}_2) + \frac{3 \ln 2}{2\pi} C_{\text{Gf}}^{(\alpha)}(\hat{u}_1, \hat{u}_2) \right] + O(1/g^2) \right\}, \quad (4.13)$$

with explicit expressions presented in Appendix C.3. The latter are the same obtained either by using gauge bound state, i.e., Eqs. (3.54), (3.55), or small fermion, Eqs. (2.55), (2.56), flux-tube functions providing a very nice check on the formalism.

Using explicit solutions, we find immediately at strong coupling

$$\ln S_{\text{Gf}}(u_1, u_2) = \frac{i\ell}{4\sqrt{2}g(\hat{u}_1 - \hat{u}_2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_2} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \sqrt{2} \right], \quad (4.14)$$

$$\ln S_{*\text{Gf}}(u_1, u_2) = \frac{\ell}{4\sqrt{2}g(\hat{u}_1 - \hat{u}_2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + i\sqrt{2} \right], \quad (4.15)$$

confirming leading order results of Ref. [20]. With Eq. (4.13), we can now uncover subleading terms.

The pentagon transitions at strong coupling are computed making use of the formulas derived in Appendix B.2. For the gauge bound states transitioning into small fermion (and vice versa), the results are

$$P_{\text{Gf}}(u_1|u_2) = (-1)^\ell \frac{\hat{u}_1 - \hat{u}_2 - i\frac{\ell}{4g}}{\hat{u}_1 - \hat{u}_2 + i\frac{\ell}{4g}} \left[\frac{\hat{x}[\hat{u}_2] - \hat{x}^{[-\ell]}[\hat{u}_1]}{\hat{x}[\hat{u}_2] - \hat{x}^{[+\ell]}[\hat{u}_1]} \right]^{1/2} \quad (4.16)$$

$$\begin{aligned}
& \times \exp \left(-i f_{\text{Gf}}^{(1)}(u_1, u_2) + i f_{\text{Gf}}^{(2)}(u_1, u_2) - f_{\text{Gf}}^{(3)}(u_1, u_2) + f_{\text{Gf}}^{(4)}(u_1, u_2) \right), \\
P_{\text{f}|\text{G}}(u_2|u_1) &= i(-1)^\ell \left[\frac{\hat{x}[\hat{u}_2] - \hat{x}^{[-\ell]}[\hat{u}_1]}{\hat{x}[\hat{u}_2] - \hat{x}^{[+\ell]}[\hat{u}_1]} \right]^{1/2} \\
& \times \exp \left(i f_{\text{Gf}}^{(1)}(u_1, u_2) - i f_{\text{Gf}}^{(2)}(u_1, u_2) - f_{\text{Gf}}^{(3)}(u_1, u_2) + f_{\text{Gf}}^{(4)}(u_1, u_2) \right),
\end{aligned} \tag{4.17}$$

and consist in substituting the phases from Appendix C.3 along with Taylor expanding prefactors following the conventions introduced in Eqs. (2.63) and (3.60), for rescaled small fermion and gauge Zhukowski variables, respectively.

5 Constraints from Descent Equation

Before we turn to applications, let us provide an additional layer of constraints on the form of the strong-coupling expansion for pentagons. This is offered by the Descent Equation [50, 51] which was recently studied within the context of the pentagon OPE in Ref. [52].

For the fermion-fermion pentagon, one can immediately find, making use of the results derived in Sect. 2, that it verifies the condition derived in [52] when one passes to the small fermion kinematics which dominates the strong coupling limit,

$$\int \frac{d\hat{x}_{\text{f}}}{\hat{x}_{\text{f}}} (1 - \hat{x}_{\text{f}}^2) \mu_{\text{f}}(u) e^{-\tau'[E_{\text{f}}(u)-1]} \delta(p_{\text{f}}(u)) P_{\text{f}|\text{f}}(-u|v) = \frac{4i}{\Gamma(g)}, \tag{5.1}$$

with the right-hand side defined by the cusp anomalous dimension that admits the following strong-coupling expansion [38, 39, 40, 41, 42, 30, 43, 44, 45, 46, 48, 49]

$$\Gamma(g) = 2g - \frac{3 \ln 2}{2\pi} + O(1/g). \tag{5.2}$$

Above, we changed from the rapidity variable \hat{u} to the Zhukowski \hat{x}_{f} via $\hat{u} = (\hat{x}_{\text{f}} + \hat{x}_{\text{f}}^{-1})/2$ in the integration measure and adopted the small fermion energy and momentum dispersion relation from Ref. [9]

$$E_{\text{f}}(u) = 1 + O(\hat{x}_{\text{f}}^2), \quad p_{\text{f}}(u) = \frac{\Gamma(g)}{2g} \hat{x}_{\text{f}} + O(\hat{x}_{\text{f}}^3). \tag{5.3}$$

Another check involves the fermion-gluon pentagon, see Eq. (39) in Ref. [52]. Passing in that relation to the fermion and Goldstone sheets for fermions and gauge excitations, respectively, we find

$$\begin{aligned}
& \int \frac{d\hat{x}_{\text{f}}}{\hat{x}_{\text{f}}} (1 - \hat{x}_{\text{f}}^2) \mu_{\text{f}}(u) e^{-\tau'[E_{\text{f}}(u)-1]} \delta(p_{\text{f}}(u)) \int d\mu_{\text{G}}(v) \left[P_{\text{f}|\text{G}}(-u|v) \left[\frac{\hat{x}^+[\hat{v}]}{\hat{x}^-[\hat{v}]} \right]^{1/2} - i \right] \\
&= -\frac{2ig^2}{\Gamma(g)} \int d\mu_{\text{G}}(v) \left[\frac{g}{\hat{x}^-[\hat{v}]} - \frac{g}{\hat{x}^+[\hat{v}]} - \frac{i}{2} (E_{\text{G}}(v) + ip_{\text{G}}(v)) \right],
\end{aligned} \tag{5.4}$$

where we introduced a differential of the integration measure for later convenience that includes the propagating “phase” factor

$$d\mu_{\text{p}}(v) = \frac{dv}{2\pi} \mu_{\text{p}}(v) e^{-\tau E_{\text{p}}(v) + i\sigma p_{\text{p}}(v)}, \tag{5.5}$$

with $p = G$ for the case at hand. A simple counting of powers of the 't Hooft coupling immediately exhibits the fact that this equation relates contributions at different orders in g^2 , i.e., its left-hand side requires effects an order higher in coupling compared to its right-hand side. Using the explicit strong coupling solutions from the previous section (for $\ell = 1$), we can expand the left-hand side in the vicinity of $\hat{x}_f = 0$,

$$P_{f|G}(-u|v) \left[\frac{\hat{x}^+[v]}{\hat{x}^-[v]} \right]^{1/2} = i + \frac{i\hat{x}_f}{2g} \left[\frac{g}{\hat{x}^-[v]} - \frac{g}{\hat{x}^+[v]} - \frac{i}{2}(E_G(v) + ip_G(v)) \right] + O(\hat{x}_f^2), \quad (5.6)$$

reproducing the one on the right. Here the energy and momentum of a single gauge excitation are [9]

$$E_G(v) \simeq \frac{1}{\sqrt{2}} \left[\left(\frac{1+\hat{v}}{1-\hat{v}} \right)^{1/4} + \left(\frac{1-\hat{v}}{1+\hat{v}} \right)^{1/4} \right], \quad p_G(v) \simeq \frac{1}{\sqrt{2}} \left[\left(\frac{1+\hat{v}}{1-\hat{v}} \right)^{1/4} - \left(\frac{1-\hat{v}}{1+\hat{v}} \right)^{1/4} \right], \quad (5.7)$$

at leading order, with subleading terms in coupling which can be extracted from Appendix D.2 of Ref. [9].

6 Application

As an immediate application of the just derived strong-coupling results, we consider the $\chi_1\chi_4^3$ component $\mathcal{W}_6^{(\chi_1\chi_4^3)}$ of the NMHV hexagon,—a function of three conformal cross ratios τ, σ, ϕ ,—in the OPE limit $\tau \rightarrow \infty$. Though we systematically constructed the $1/g$ expansion in the previous sections, we will restrict our consideration below to leading effects in g only in order to observe the emergence of the classical string area from the summation of the pentagon OPE series. The study of subleading terms is much more cumbersome and is postponed to a future study.

We start our analysis with the consideration of the contribution of the fermion, that carries the Grassmann quantum numbers of the $\mathcal{W}_6^{(\chi_1\chi_4^3)}$ component of the hexagon, along with the bound state of ℓ gluons, i.e., the states $|\ell(u)f(v)\rangle$. Thus, we have to resum the series

$$\mathcal{W}_6^{(\chi_1\chi_4^3)} = \sum_{\ell=1}^{\infty} e^{i(\ell+1/2)\phi} \mathcal{W}_{\ell f} \quad (6.1)$$

where the individual contributions admit the form

$$\mathcal{W}_{\ell f} = \int_{C_f} \int_{C_G} \frac{d\mu_G(u) d\mu_f(v) (-i)x_f[v]}{|P_{G|f}(u|v)|^2}. \quad (6.2)$$

To make notations in the integrand more compact, here and below $|P_{p|p'}(u|v)|^2$ stands for $|P_{p|p'}(u|v)|^2 = P_{p|p'}(u|v)P_{p'|p}(v|u)$. Above, the differential measures were introduced in Eq. (5.5) and the integration contour for the small fermion is $C_f = (-\infty, -2g) \cup (2g, \infty)$. For the gluon it is bound to the interval $C_G = (-2g, 2g)$, since outside of it the gauge excitation behaves as a giant hole, i.e., its energy and momentum scale as a first power of 't Hooft coupling g , and induce only exponentially suppressed contribution to the Wilson loop. By virtue of the complementarity of the fermionic and gluonic domains, we cannot hit the pole in (4.16) at strong coupling.

Then, at leading order in strong coupling $|P_{\text{f|G}}|^2 \sim 1$ and the integral over rapidities factorizes by virtue of this property. Therefore, the sum over all values of ℓ in Eq. (6.1) can be evaluated in a closed form⁵,

$$\mathcal{W}_6^{(\chi_1 \chi_4^3)} = e^{i\phi/2} \int d\mu_{\text{f}}(v)(-i)x_{\text{f}}[v] \left(1 - \int \frac{du}{2\pi} \mu_{\text{G}}(u) \text{Li}_2 \left(e^{-\tau E_{\text{G}}(u) + i\sigma p_{\text{G}}(u) + i\phi} \right) \right). \quad (6.3)$$

The second term in braces is of order g and is the first term in the expansion of the exponential of the minimal area. To restore the latter, one has to resum all one-fermion–multiple gauge bound states contributions. For N of these bound states accompanying the fermion, we find

$$\frac{e^{i\phi/2}}{N!} \sum_{\ell_1, \dots, \ell_N=1}^{\infty} e^{i\phi(\ell_1 + \dots + \ell_N)} \int d\mu_{\text{f}}(v)(-i)x_{\text{f}}[v] \int \frac{d\mu_{\text{G}}(u_1) \dots d\mu_{\text{G}}(u_N)}{\prod_{j=1}^N |P_{\text{G|f}}(u_j|v)|^2 \prod_{k>j=1}^N |P_{\text{G|G}}(u_k|u_j)|^2}. \quad (6.4)$$

Again by virtue of the scaling $P_{\text{G|G}} \sim 1$, we observe factorization and, after the summation over N , we deduce

$$\mathcal{W}_6^{(\chi_1 \chi_4^3)} = e^{i\phi/2} \int d\mu_{\text{f}}(v)(-i)x_{\text{f}}[v] \exp \left(- \int \frac{du}{2\pi} \mu_{\text{G}}(u) \text{Li}_2 \left(e^{-\tau E_{\text{G}}(u) + i\sigma p_{\text{G}}(u) + i\phi} \right) \right). \quad (6.5)$$

Let us clarify that here and in Eq. (6.3), μ_{G} stands for the single-gluon measure. Adding to this the effect of antigluon bound states, we modify the exponent by an addendum that differs from the displayed term by a mere sign change in front of ϕ . In this manner, we recover the gluon portion of the minimal area in the $\tau \rightarrow \infty$ limit of the NMHV amplitude, which obviously contains an overall factor of integrated fermionic measure that is responsible for quantum numbers of the component of the superWilson loop under study.

The contribution to MHV amplitude at strong coupling receives an extra effect from an elusive excitation of mass two [3, 27, 28]. As was first explained in Ref. [12] within the OPE framework, it is not an elementary but rather a virtual composite state of small fermion-antifermion pair that comes on-shell as a bound state at infinite coupling. This idea was further pursued in an effective framework of Ref. [20] that assumed the existence of bound states of these mesons to reproduce the result from Thermodynamic Bethe Ansatz [3, 27, 28].

For the case at hand, we thus continue with the contribution of $|\bar{\text{f}}(u_1)\text{f}(v_1)\text{f}(v_2)\rangle$ state to the NMHV hexagon

$$\mathcal{W}_{(\bar{\text{f}}\text{f})\text{f}} = \frac{1}{2!1!} \int du_1 dv_1 dv_2 \frac{\mu_{\text{f}}(u_1)\mu_{\text{f}}(v_1)\mu_{\text{f}}(v_2)}{|P_{\text{f}|\bar{\text{f}}}(u_1|v_1)P_{\text{f}|\bar{\text{f}}}(u_1|v_2)P_{\text{f}|\text{f}}(v_1|v_2)|^2} \frac{x_{\text{f}}[v_1]x_{\text{f}}[v_2]}{x_{\text{f}}[u_1]} \mathcal{R}_1(u_1, v_1, v_2), \quad (6.6)$$

where \mathcal{R}_1 is a matrix part of the transition. The form of the latter for any internal symmetry group quantum numbers was recently worked out in Ref. [53]. What is important for the current analysis is that it has the following generic form

$$\mathcal{R}_N(\mathbf{u}_N, \mathbf{v}_{N+1}) = \frac{P_N(\mathbf{u}_N, \mathbf{v}_{N+1})}{\prod_{j>i} [(v_j - v_i)^2 + 1] \prod_{l>k} [(u_l - u_k)^2 + 1] \prod_{m,n} [(u_m - v_n)^2 + 4]}, \quad (6.7)$$

⁵Here we employed the well-known series representation of the dilogarithm $\text{Li}_2(z) = \sum_{\ell=1}^{\infty} z^{\ell}/\ell^2$. Let us point out that comparing the obtained expression with Eq. (F.46) of Ref. [2], one has to realize that the parameter μ in this reference is related to the angle ϕ via $\mu = -e^{i\phi}$ as stated after Eq. (F.51). So the argument of the dilogarithm comes with a plus sign.

and possesses poles expected for nonsinglet transitions [12, 15]. The polynomial in the numerator is of degree 2^{2N-1} in variables $\mathbf{u}_N = (u_1, \dots, u_N)$ and $\mathbf{v}_{N+1} = (v_1, \dots, v_{N+1})$. The lowest nontrivial one is

$$P_1(u_1, v_1, v_2) = 40 + 6u_1^2 + 4v_1^2 - 2v_1v_2 + 4v_2^2 - 6u_1(v_1 + v_2). \quad (6.8)$$

Rescaling the fermionic rapidities with the coupling constant, $u_j = 2g\hat{u}_j$ etc., one observes that the $\mathcal{W}_{(\bar{\text{f}}\text{f})\text{f}}$ would be suppressed compared to the contribution of gluons analyzed above. However, there is a subtlety here, pointed out in Ref. [12], that the integration contour gets pinched by the aforementioned poles as $g \rightarrow \infty$. Thus one has to move the integration contour for u_1 to the lower half-plane picking up two poles along the way $u_1 = v_j - 2i$ ($j = 1, 2$). The latter induce leading order effect in coupling, on the same footing as gauge fields, and read

$$\mathcal{W}_{(\bar{\text{f}}\text{f})\text{f}} = \int d\mu_{\text{f}}(v_2)(-i)x_{\text{f}}[v_2] \int d\mu_{\bar{\text{f}}\text{f}}(v_1) \frac{x_{\text{f}}^{[+2]}[v_1]}{x_{\text{f}}^{[-2]}[v_1]} \frac{-1}{|P_{\text{f}|\bar{\text{f}}}(v_1^{[-2]}|v_2)|^2 |P_{\text{f}|\text{f}}(v_1^{[+2]}|v_2)|^2 (v_1 - v_2)(v_1^{[+2]} - v_2)}, \quad (6.9)$$

where we dropped subleading contributions from the deformed contour. Here the composite fermion-antifermion measure is [12]

$$\mu_{\bar{\text{f}}\text{f}}(v) = -\frac{\mu_{\text{f}}(v+i)\mu_{\text{f}}(v-i)}{|P_{\text{f}|\bar{\text{f}}}(v+i|v-i)|^2}. \quad (6.10)$$

with the energy/momentum of the composite excitation being $E_{\bar{\text{f}}\text{f}}(v) = E_{\text{f}}(v+i) + E_{\text{f}}(v-i)/p_{\bar{\text{f}}\text{f}}(v) = p_{\text{f}}(v+i) + p_{\text{f}}(v-i)$. Making use of the explicit expressions for the pentagons at strong coupling, one finds that the expression accompanying measures in Eq. (6.9) goes to minus one at leading order, yielding a product representation of the single fermion accompanied by the $(\bar{\text{f}}\text{f})$ -pair propagating in the OPE channel.

Generally, for N $(\bar{\text{f}}\text{f})$ -pairs, we have

$$\begin{aligned} \mathcal{W}_{(\bar{\text{f}}\text{f})^{N_{\text{f}}}} &= \frac{1}{N!(N-1)!} \int \frac{\prod_{i=1}^N d\mu_{\text{f}}(u_i) \prod_{j=1}^{N+1} d\mu_{\text{f}}(v_j)}{|\prod_{j>i} P_{\text{f}|\text{f}}(v_j|v_i)^2 \prod_{l>k} P_{\text{f}|\text{f}}(u_l|u_k) \prod_{m,n} P_{\text{f}|\bar{\text{f}}}(u_m|v_n)|^2} \\ &\quad \times \left(\prod_{i=1}^N \frac{x_{\text{f}}[v_i]}{x_{\text{f}}[u_i]} \right) (-i)x_{\text{f}}[v_{N+1}] \mathcal{R}_N(\mathbf{u}_N, \mathbf{v}_{N+1}). \end{aligned} \quad (6.11)$$

The polynomial \mathcal{R}_N obeys a very important property: taking the residue of \mathcal{R}_N , for instance, at $u_1 = v_1 + 2i$, yields

$$\begin{aligned} \text{res}_{u_1=v_1+2i} \mathcal{R}_N(\mathbf{u}_N, \mathbf{v}_{N+1}) &= \frac{1}{\prod_{j>1} [v_j - v_1][v_j - v_1 - i] \prod_{k>1} [u_k - v_1 - 2i][u_k - v_1 - i]} \\ &\quad \times \frac{P_{N-1}(\mathbf{u}_{N-1}, \mathbf{v}_N)}{\prod_{j>i>1} [(v_j - v_i)^2 + 1] \prod_{l>k>1} [(u_l - u_k)^2 + 1] \prod_{m,n \neq 1} [(u_m - v_n)^2 + 4]}, \end{aligned} \quad (6.12)$$

with the polynomial of a lower degree. Thus, we do not need the explicit form of \mathcal{R}_N here. Consecutively taking the residues, we find

$$\mathcal{W}_{(\bar{\text{f}}\text{f})^{N_{\text{f}}}} = \frac{(-1)^N}{N!} \int d\mu_{\text{f}}(v_{N+1})(-i)x_{\text{f}}[v_{N+1}] \int \prod_{j=1}^N d\mu_{\bar{\text{f}}\text{f}}(v_j) \frac{x_{\text{f}}^{[+2]}[v_j]}{x_{\text{f}}^{[-2]}[v_j]} \frac{1}{\prod_{N+1>n \neq m} |P_{\text{f}|\bar{\text{f}}}(v_m + 2i|v_n)|^2}$$

$$\begin{aligned}
& \times \frac{1}{\prod_{N+1>j>i} (v_j - v_i)(v_j - v_i - i)|P_{f|f}(v_i|v_j)|^2} \frac{1}{\prod_{N>j>i} (v_j - v_i)(v_j - v_i + i)|P_{f|f}(v_i + 2i|v_j + 2i)|^2} \\
& + \dots
\end{aligned} \tag{6.13}$$

Here the displayed expression is responsible for the exponentiation of the $(\bar{f}f)$ -pair exchange. The ellipsis stand for effect of other poles which induce terms proportional to lesser powers of the composite measure (6.10). The solution of this combinatorial problem yields contributions corresponding to scattering of fermion bound states [20] which together with single pair propagating in the OPE channel results in dilogarithm expected from Thermodynamic Bethe Ansatz [3, 27, 28] at leading order in strong coupling,

$$\mathcal{W}_6^{(\chi_1\chi_4^3)} = e^{i\phi/2} \int d\mu_f(v)(-i)x_f[v] \exp \left(\int \frac{du}{2\pi} \mu_{\bar{f}\bar{f}}(u) \text{Li}_2 \left(-e^{-\tau E_{\bar{f}\bar{f}}(u) + i\sigma p_{\bar{f}\bar{f}}(u)} \right) \right). \tag{6.14}$$

In a similar fashion, one can work out mixed terms with both fermionic pairs and gluon bound states. The outcome of this consideration is that the complete leading order result is given by a single exponent with the argument determined by the sum of individual contributions discussed above. A detailed consideration is deferred to a future publication.

7 Conclusions

In this paper we initiated a systematic study of the strong coupling expansion for pentagon transitions in the OPE approach to the null polygonal superWilson loop. The framework is a generalization of a previous consideration [30, 31] for the cusp anomalous dimension, i.e., the vacuum energy density of the flux tube. While we addressed $1/g$ perturbative series, we did not include exponentially suppressed contributions in our analysis. These can be recovered in a straightforward fashion from explicit all-order representation of the flux-tube functions for relevant excitations. Presently, we considered gauge-field bound states and fermions. Their flux-tube functions can be used to find all other pentagon transitions (to complete the list of the ones explicitly given in the main text) in the perturbative string regime by means of exchange relations except the one for the hole transitions which require a separate calculation. The contribution of the latter was not addressed here with the focus being rather on the emergence of the minimal area in NMHV amplitudes. It was argued in Ref. [19] that all multi-scalar exchanges have to be resummed and were shown to induce kinematic-independent leading order effects in addition to the area for MHV case. For NMHV case, this question was recently addressed in Ref. [54]. We demonstrated there the factorization of contributions of near-massless scalars from the helicity-dependent massive particles carrying the quantum numbers of Grassmann components in question of the superWilson loop and provided a concise formula for their resummed short-distance behavior.

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A Special functions and integrals

In the body of the paper, we introduced the following special functions. The function W is related to the hypergeometric function of two variables Φ_1 [55] and reads

$$W(z, u) = \frac{\sqrt{2}}{\pi} \int_{-1}^1 dk \left(\frac{1+k}{1-k} \right)^{1/4} e^{zk} \frac{\mathcal{P}}{k-u}. \quad (\text{A.1})$$

While V and U are related to the confluent hypergeometric function of the second kind and admit the following integral representations [31]

$$V_n(z) = \frac{\sqrt{2}}{\pi} \int_{-1}^1 dk \left(\frac{1+k}{1-k} \right)^{1/4} \frac{e^{kz}}{(1+k)^n}, \quad (\text{A.2})$$

$$U_n^\pm(z) = \frac{1}{2} \int_1^\infty dk \left(\frac{k+1}{k-1} \right)^{\mp 1/4} \frac{e^{-k(z-1)}}{(k \mp 1)^n}. \quad (\text{A.3})$$

Depending on the sign of z , these functions develop different asymptotic behavior at $z \rightarrow \pm\infty$. Up to exponentially suppressed contributions, the power series in $1/z$ can be constructed from the following integrals,

$$W(\pm|z|, u)|_{|z| \rightarrow \infty} \simeq \pm \frac{e^{|z|}(2|z|)^{\pm 1/4-1}}{2^{-3/2}\pi(1 \mp u)} \int_0^\infty d\beta e^{-\beta} \beta^{\mp 1/4} \left(1 - \frac{\beta}{2|\tau|} \right)^{\pm 1/4} \left(1 - \frac{\beta}{(1 \mp u)|z|} \right)^{-1}, \quad (\text{A.4})$$

$$V_n(\pm|z|)|_{|z| \rightarrow \infty} \simeq \frac{e^{|z|}(2|z|)^{\pm(1/4-n/2)-1+n/2}}{2^{n-3/2}\pi} \int_0^\infty d\beta e^{-\beta} \beta^{\mp(1/4-n/2)-n/2} \left(1 - \frac{\beta}{2|z|} \right)^{\pm(1/4-n/2)-n/2}, \quad (\text{A.5})$$

obtained from above by a simple transformation of the integration variable. Similarly an equivalent representation for U_n ($n = 0, 1$) reads,

$$U_0^\pm(z) = (2z)^{-(4\pm 1)/4} \int_0^\infty d\beta e^{-\beta} \beta^{\pm 1/4} \left(1 + \frac{\beta}{2z} \right)^{\mp 1/4}, \quad (\text{A.6})$$

$$U_1^\pm(z) = \frac{1}{2} (2z)^{-(2\mp 1)/4} \int_0^\infty d\beta e^{-\beta} \beta^{-(2\pm 1)/4} \left(1 + \frac{\beta}{2z} \right)^{-(2\mp 1)/4}. \quad (\text{A.7})$$

Explicitly, one finds

$$\frac{W(\pm|z|, u)}{V_0(\pm|z|)} \Big|_{|z| \rightarrow \infty} = -\frac{1}{u \mp 1} \pm \frac{(4 \mp 1)}{4z(u \mp 1)^2} + O(1/z^2), \quad (\text{A.8})$$

$$\frac{V_1(\pm|z|)}{V_0(\pm|z|)} \Big|_{|z| \rightarrow \infty} = -2(1 \mp 1)|z| \pm \frac{1}{2} + \frac{4 \mp 1}{16|z|} + O(1/z^2), \quad (\text{A.9})$$

and

$$U_0^\pm(z)|_{z \rightarrow \infty} = (2z)^{-(4\pm 1)/4} \Gamma\left(\frac{4 \pm 1}{4}\right) \left[1 \mp \frac{4 \pm 1}{32z} + O(1/z^2) \right], \quad (\text{A.10})$$

$$U_1^\pm(z)|_{z \rightarrow \infty} = (2z)^{-(2 \mp 1)/4} \Gamma\left(\frac{2 \mp 1}{4}\right) \left[1 \pm \frac{4 \pm 5}{32z} + O(1/z^2)\right], \quad (\text{A.11})$$

with subleading terms eagerly evaluated from Eqs. (A.4), (A.5) and (A.6), (A.7) by Taylor expanding the integrand and computing the resulting integrals using the definition of the Euler Gamma function.

In the main text, we also introduced different parity components of V and W for the imaginary value of their argument $z = -i\tau$. They are $W(-i\tau, u) = W^+(\tau, u) - iW^-(\tau, u)$,

$$\begin{aligned} W^+(\tau, u) &= \frac{\sqrt{2}}{\pi} \int_{-1}^1 dk \left(\frac{1+k}{1-k}\right)^{1/4} \cos(\tau k) \frac{\mathcal{P}}{k-u}, \\ W^-(\tau, u) &= \frac{\sqrt{2}}{\pi} \int_{-1}^1 dk \left(\frac{1+k}{1-k}\right)^{1/4} \sin(\tau k) \frac{\mathcal{P}}{k-u}, \end{aligned} \quad (\text{A.12})$$

and $V_n(-i\tau) = V_n^+(\tau) - iV_n^-(\tau)$,

$$\begin{aligned} V_n^+(\tau) &= \frac{\sqrt{2}}{\pi} \int_{-1}^1 dk \left(\frac{1+k}{1-k}\right)^{1/4} \frac{\cos(\tau k)}{(1+k)^n}, \\ V_n^-(\tau) &= \frac{\sqrt{2}}{\pi} \int_{-1}^1 dk \left(\frac{1+k}{1-k}\right)^{1/4} \frac{\sin(\tau k)}{(1+k)^n}. \end{aligned} \quad (\text{A.13})$$

Finally, the only two integrals that are needed to solve the singular integral equations as well to derive the explicit expressions for all dynamical phases quoted below in Appendix C are the following

$$\int_{-1}^1 \frac{dk}{\pi} \left(\frac{1-k}{1+k}\right)^{1/4} \frac{1}{k-q} = \sqrt{2} \left(\frac{q-1}{q+1}\right)^{1/4} - \sqrt{2}, \quad (\text{A.14})$$

$$\int_{-1}^1 \frac{dk}{\pi} \left(\frac{1-k}{1+k}\right)^{1/4} \frac{\mathcal{P}}{k-p} = \left(\frac{1-p}{1+p}\right)^{1/4} - \sqrt{2}. \quad (\text{A.15})$$

These are valid for $|q| > 1$ and $|p| < 1$, respectively,

B Gauge pentagons: to Goldstone sheet and back

In this appendix we will construct pentagons for gauge field bound states. The initial point of this consideration is transitions for a single gluon undergoing a transformation into the same or another flux-tube excitation. On the physical sheet, the analytical properties of the flux-tube functions are quite complex due to the presence of an infinite number of cuts $[-2g, 2g]$ that are equidistantly separated along the imaginary axis starting at $\Im m[u] = \pm \frac{1}{2}$ and going to infinity. So it becomes problematic to construct the bound state observables by fusing single-particle once staying in the kinematical region of rapidities $-2g < u < 2g$. A way out of this complication is to make an analytic continuation to the Goldstone (or half-mirror) sheet which has just two cuts for the gauge field at $\Im m[u] = \pm \frac{1}{2}$ [36]. Therefore, as proposed in Ref. [14], for technical and practical reasons it is instructive to move upwards through the first cut of the gluon excitation to the half-mirror (Goldstone) sheet

$$u \rightarrow u^G = u + i\frac{\ell}{2} + i0 \rightarrow u, \quad (\text{B.1})$$

(this results in the change $x^-[u] \rightarrow g^2/x^-[u]$ while $x^+[u]$ stays intact) and fuse elementary excitations there, keeping the imaginary part of their rapidities above the cut, i.e., $\Im m[u] > \frac{1}{2}$. Once ℓ of these gluons are fused together, one can always move back to the physical sheet, now passing to it through the top cut of the bound state $[-2g + i\frac{\ell}{2}, 2g + i\frac{\ell}{2}]$. This implies that the Zhukowsky variables obeys the following transformation rules

$$x^{[-\ell]}[u] \rightarrow \frac{g^2}{x^{[-\ell]}[u]}, \quad x^{[+\ell]}[u] \rightarrow x^{[+\ell]}[u]. \quad (\text{B.2})$$

B.1 Bound-state-bound-state pentagons

We start with the gluon-gluon and gluon-antigluon pentagons. These are given as usual by the ratio of the direct and mirror S-matrices (see Eqs. (3.1) and (3.2) of the main text where one sets $\ell = 1$) [1, 10]

$$P_{\text{g|g}}^2(u_1|u_2) = w_{\text{gg}}^{-1}(u_1, u_2) \frac{S_{\text{gg}}(u_1, u_2)}{S_{*\text{gg}}(u_1, u_2)}, \quad P_{\text{g|\bar{g}}}^2(u_1|u_2) = w_{\text{gg}}(u_1, u_2) \frac{S_{\text{g\bar{g}}}(u_1, u_2)}{S_{*\text{g\bar{g}}}(u_1, u_2)}, \quad (\text{B.3})$$

with the prefactor being

$$w_{\text{gg}}(u_1, u_2) = \frac{g^2(u_1 - u_2)(u_1 - u_2 - i)}{x^+[u_1]x^-[u_1]x^+[u_2]x^-[u_2]} \left(1 - \frac{g^2}{x^+[u_1]x^-[u_2]}\right)^{-1} \times \left(1 - \frac{g^2}{x^-[u_1]x^+[u_2]}\right)^{-1} \left(1 - \frac{g^2}{x^+[u_1]x^+[u_2]}\right)^{-1} \left(1 - \frac{g^2}{x^-[u_1]x^-[u_2]}\right)^{-1}. \quad (\text{B.4})$$

Going to the Goldstone sheet, we find the latter changes to

$$w_{\text{GG}}(u_1, u_2) = \frac{u_1 - u_2}{u_1 - u_2 + i} \frac{\left(1 - \frac{g^2}{x^+[u_1]x^-[u_2]}\right) \left(1 - \frac{g^2}{x^-[u_1]x^+[u_2]}\right)}{\left(1 - \frac{g^2}{x^+[u_1]x^+[u_2]}\right) \left(1 - \frac{g^2}{x^-[u_1]x^-[u_2]}\right)}, \quad (\text{B.5})$$

and the S-matrices turn into Eqs. (3.16) and (3.17) with $\ell = 1$, respectively. Now, as explained in the preamble, the fusion is straightforward on this sheet as one is away from all the cuts and obtains

$$w_{\text{GG}}(u_1, u_2) = \prod_{k_1=1}^{\ell_1} \prod_{k_2=1}^{\ell_2} w_{\text{GG}}(u_1^{[2k_1-\ell_1-1]}, u_2^{[2k_2-\ell_2-1]}) = s_{*\ell_1\bar{\ell}_2}(u_1, u_2) \frac{\left(1 - \frac{g^2}{x^{[\ell_1]}[u_1]x^{[-\ell_2]}[u_2]}\right) \left(1 - \frac{g^2}{x^{[-\ell_1]}[u_1]x^{[\ell_2]}[u_2]}\right)}{\left(1 - \frac{g^2}{x^{[-\ell_1]}[u_1]x^{[-\ell_2]}[u_2]}\right) \left(1 - \frac{g^2}{x^{[\ell_1]}[u_1]x^{[\ell_2]}[u_2]}\right)}. \quad (\text{B.6})$$

Here $\Im m[u_k] > \ell_k/2$ ($k = 1, 2$). We obviously abused notations in the first line by calling the single gluon and bound state dressing factors by the same symbol. It will be always clear from the context what we are dealing with.

Passing to the physical sheet, but now through the top Zhukowski cut of ℓ -bound state, i.e., $x^{[-\ell_k]}[u_k] \rightarrow g^2/x^{[-\ell_k]}[u_k]$, we find

$$w_{\ell_1\ell_2}(u_1, u_2) = s_{*\ell_1\bar{\ell}_2}(u_1, u_2) \frac{g^2 \left((u_1 - u_2)^2 + \frac{(\ell_1 + \ell_2)^2}{4} \right)}{x^{[\ell_1]}[u_1]x^{[-\ell_1]}[u_1]x^{[\ell_2]}[u_2]x^{[-\ell_2]}[u_2]} \left(1 - \frac{g^2}{x^{[\ell_1]}[u_1]x^{[-\ell_2]}[u_2]}\right)^{-1} \quad (\text{B.7})$$

$$\times \left(1 - \frac{g^2}{x^{[-\ell_1]}[u_1]x^{[\ell_2]}[u_2]}\right)^{-1} \left(1 - \frac{g^2}{x^{[-\ell_1]}[u_1]x^{[-\ell_2]}[u_2]}\right)^{-1} \left(1 - \frac{g^2}{x^{[\ell_1]}[u_1]x^{[\ell_2]}[u_2]}\right)^{-1}.$$

which determines the gauge stack-(anti)stack pentagons when it is accompanied by the ratio of bound-state S-matrices (3.1) and (3.2)

$$P_{\ell_1|\ell_2}^2(u_1|u_2) = w_{\ell_1\ell_2}^{-1}(u_1, u_2) \frac{S_{\ell_1\ell_2}(u_1, u_2)}{S_{*\ell_1\ell_2}(u_1, u_2)}, \quad P_{\ell_1|\bar{\ell}_2}^2(u_1|u_2) = w_{\ell_1\bar{\ell}_2}(u_1, u_2) \frac{S_{\ell_1\bar{\ell}_2}(u_1, u_2)}{S_{*\ell_1\bar{\ell}_2}(u_1, u_2)}. \quad (\text{B.8})$$

To compare with known results, let us give them in the explicit form. Using the relation

$$\begin{aligned} & \exp\left(2i\sigma_{\ell_1\ell_2}(u_1, u_2) - 2\widehat{\sigma}_{\ell_1\ell_2}(u_1, u_2)\right) \\ &= \frac{\Gamma^2\left(1 + \frac{\ell_1+\ell_2}{2} + iu_1 - iu_2\right)}{\Gamma^2\left(1 + \frac{\ell_1}{2} + iu_1\right)\Gamma^2\left(1 + \frac{\ell_2}{2} - iu_2\right)} \frac{x^{[\ell_1]}[u_1]x^{[-\ell_1]}[u_1]x^{[\ell_2]}[u_2]x^{[-\ell_2]}[u_2]}{g^2\left((u_1 - u_2)^2 + \frac{(\ell_1+\ell_2)^2}{4}\right)} \\ & \times \exp\left(2\int_0^\infty \frac{dt}{t(e^t - 1)} (J_0(2gt) - 1) (J_0(2gt) + 1 - e^{-iu_1t - \ell_1t/2} - e^{iu_2t - \ell_2t/2})\right), \end{aligned} \quad (\text{B.9})$$

we can cast the helicity-violating pentagon in the form

$$\begin{aligned} P_{\ell_1|\bar{\ell}_2}(u_1|u_2) &= \frac{\Gamma\left(1 + \frac{\ell_1+\ell_2}{2} + iu_1 - iu_2\right)}{\Gamma\left(1 + \frac{\ell_1}{2} + iu_1\right)\Gamma\left(1 + \frac{\ell_2}{2} - iu_2\right)} \left(1 - \frac{g^2}{x_1^{[\ell_1]}x_2^{[-\ell_2]}}\right)^{-1/2} \\ & \times \left(1 - \frac{g^2}{x_1^{[-\ell_1]}x_2^{[\ell_2]}}\right)^{-1/2} \left(1 - \frac{g^2}{x_1^{[-\ell_1]}x_2^{[-\ell_2]}}\right)^{-1/2} \left(1 - \frac{g^2}{x_1^{[\ell_1]}x_2^{[\ell_2]}}\right)^{-1/2} \\ & \times \exp\left(\int_0^\infty \frac{dt}{t(e^t - 1)} (J_0(2gt) - 1) (J_0(2gt) + 1 - e^{-iu_1t - \ell_1t/2} - e^{iu_2t - \ell_2t/2})\right) \\ & \times \exp\left(-if_{\ell_1\ell_2}^{(1)}(u_1, u_2) + if_{\ell_1\ell_2}^{(2)}(u_1, u_2) + f_{\ell_1\ell_2}^{(3)}(u_1, u_2) - f_{\ell_1\ell_2}^{(4)}(u_1, u_2)\right). \end{aligned} \quad (\text{B.10})$$

Here the dynamical phases are given in the text in Eqs. (3.6), (3.7), (3.9) and (3.10). While making use of the relation

$$s_{\ell_1\ell_2}(u_1, u_2) = \frac{s_{*\ell_2\bar{\ell}_1}(u_2, u_1)}{s_{*\ell_1\bar{\ell}_2}(u_1, u_2)} \quad (\text{B.11})$$

we can take the square of the right-hand side of Eq. (B.8) to find

$$\begin{aligned} P_{\ell_1|\ell_2}(u_1|u_2) &= \frac{(-1)^{\ell_2}\Gamma\left(\frac{\ell_1+\ell_2}{2} - iu_1 + iu_2\right)\Gamma\left(\frac{\ell_1-\ell_2}{2} + iu_1 - iu_2\right)}{\Gamma\left(1 + \frac{\ell_1}{2} + iu_1\right)\Gamma\left(1 + \frac{\ell_2}{2} - iu_2\right)\Gamma\left(1 + \frac{\ell_1-\ell_2}{2} - iu_1 + iu_2\right)} \\ & \times \left(x_1^{[\ell_1]}x_2^{[-\ell_2]} - g^2\right)^{1/2} \left(x_1^{[-\ell_1]}x_2^{[\ell_2]} - g^2\right)^{1/2} \left(x_1^{[-\ell_1]}x_2^{[-\ell_2]} - g^2\right)^{1/2} \left(x_1^{[\ell_1]}x_2^{[\ell_2]} - g^2\right)^{1/2} \\ & \times \exp\left(\int_0^\infty \frac{dt}{t(e^t - 1)} (J_0(2gt) - 1) (J_0(2gt) + 1 - e^{-iu_1t - \ell_1t/2} - e^{iu_2t - \ell_2t/2})\right) \\ & \times \exp\left(-if_{\ell_1\ell_2}^{(1)}(u_1, u_2) + if_{\ell_1\ell_2}^{(2)}(u_1, u_2) + f_{\ell_1\ell_2}^{(3)}(u_1, u_2) - f_{\ell_1\ell_2}^{(4)}(u_1, u_2)\right). \end{aligned} \quad (\text{B.12})$$

Both of these expressions agree with Ref. [14].

B.2 Bound-state–fermion pentagons

Next we turn to the gauge bound-state–(anti)fermion pentagons. These are constructed from the single gauge field–(anti)fermion transitions which read [15]

$$P_{g|f}^2(u_1|u_2) = w_{gf}(u_1, u_2) \frac{S_{gf}(u_1, u_2)}{S_{*gf}(u_1, u_2)}, \quad P_{g|\bar{f}}^2(u_1|u_2) = w_{gf}^{-1}(u_1, u_2) \frac{S_{gf}(u_1, u_2)}{S_{*gf}(u_1, u_2)}, \quad (\text{B.13})$$

with

$$w_{gf}(u_1, u_2) = (u_1 - u_2 + \frac{i}{2}) \frac{x_f[u_2]}{x^+[u_1]x^-[u_1]} \left(1 - \frac{x_f[u_2]}{x^+[u_1]}\right)^{-1} \left(1 - \frac{x_f[u_2]}{x^-[u_1]}\right)^{-1}, \quad (\text{B.14})$$

and scattering matrices quoted in the body of the paper in Eqs. (4.1) and (4.1) for $\ell = 1$. Going to the Goldstone sheet, we find

$$w_{Gf}(u_1, u_2) = -\frac{u_1 - u_2 + \frac{i}{2} x^-[u_1]x[u_2] - g^2}{u_1 - u_2 - \frac{i}{2} x^+[u_1]x[u_2] - g^2}. \quad (\text{B.15})$$

The fusion of the w factor produces

$$w_{Gf}(u_1, u_2) = \prod_{k=1}^{\ell} w_{Gf}(u_1^{[2k-\ell-1]}, u_2) = (-1)^{\ell} \frac{(u_1 - u_2 + i\frac{\ell}{2}) (x^{[-\ell]}[u_1]x[u_2] - g^2)}{(u_1 - u_2 - i\frac{\ell}{2}) (x^{[+\ell]}[u_1]x[u_2] - g^2)}. \quad (\text{B.16})$$

Again, we abused the notation here by calling the bound state and single gauge field prefactors by the same letter. Going back to the physical sheet, we find

$$w_{\ell f}(u_1, u_2) = (-1)^{\ell+1} (u_1 - u_2 + i\frac{\ell}{2}) \frac{x_f[u_2]}{x^{[\ell]}[u_1]x^{[-\ell]}[u_1]} \left(1 - \frac{x_f[u_2]}{x^{[\ell]}[u_1]}\right)^{-1} \left(1 - \frac{x_f[u_2]}{x^{[-\ell]}[u_1]}\right)^{-1}. \quad (\text{B.17})$$

Analogously, for the gauge bound state–antifermion case, we get

$$w_{\ell \bar{f}}(u_1, u_2) = w_{\ell f}^{-1}(u_1, u_2). \quad (\text{B.18})$$

In this manner we derive the stack–(anti)fermion pentagons

$$P_{\ell|f}^2(u_1|u_2) = w_{\ell f}(u_1, u_2) \frac{S_{\ell f}(u_1, u_2)}{S_{*\ell f}(u_1, u_2)}, \quad P_{\ell|\bar{f}}^2(u_1|u_2) = w_{\ell f}^{-1}(u_1, u_2) \frac{S_{\ell f}(u_1, u_2)}{S_{*\ell f}(u_1, u_2)}, \quad (\text{B.19})$$

which read, respectively,

$$P_{\ell|f}(u_1|u_2) = \frac{i}{g} \frac{(u - v + i\frac{\ell}{2}) x_f[u_2]}{(x^{[+\ell]}[u_1] - x_f[u_2])^{1/2} (x^{[-\ell]}[u_1] - x_f[u_2])^{1/2}} \quad (\text{B.20})$$

$$\times \exp \left(-i f_{\ell f}^{(1)}(u_1, u_2) + i f_{\ell f}^{(2)}(u_1, u_2) + f_{\ell f}^{(3)}(u_1, u_2) - f_{\ell f}^{(4)}(u_1, u_2) \right),$$

$$P_{\ell|\bar{f}}(u_1|u_2) = i g (x^{[+\ell]}[u_1] - x_f[u_2])^{1/2} (x^{[-\ell]}[u_1] - x_f[u_2])^{1/2} \quad (\text{B.21})$$

$$\times \exp \left(-i f_{\ell f}^{(1)}(u_1, u_2) + i f_{\ell f}^{(2)}(u_1, u_2) + f_{\ell f}^{(3)}(u_1, u_2) - f_{\ell f}^{(4)}(u_1, u_2) \right),$$

with dynamical phases quoted in Eqs. (4.3) – (4.6). These expressions are in agreement with Ref. [18] up to a different choice of normalization conventions.

In the main text, we also use the pentagon with flipped flux-tube excitations, i.e., $P_{f|\ell}$, and continued to the Goldstone sheet, $P_{f|G}$. This transition can be obtained in two steps, First, one uses the fact that on the physical sheet,

$$P_{f|\ell}(u_2|u_1) = P_{\ell|f}(-u_1|-u_2). \quad (\text{B.22})$$

Then use the following obvious properties of dynamical phases

$$f_{pp'}^{(1,2)}(-u_1, -u_2) = -f_{pp'}^{(1,2)}(u_1, u_2), \quad f_{pp'}^{(3,4)}(-u_1, -u_2) = +f_{pp'}^{(3,4)}(u_1, u_2), \quad (\text{B.23})$$

and only after that continuing the gauge bound state to the Goldstone sheet. In this fashion, we find Eq. (4.17).

C Dynamical phases

In this appendix we summarize dynamical phases for fermion-fermion, gluon bound-state-bound state and fermion-gluon bound state transitions to the first nontrivial order in $1/g$. In a similar fashion, one can find the rest of transitions by means of the exchange relations, except for the hole-hole case, which requires a separate study.

C.1 Fermion-fermion case

For the fermion-fermion phases, the leading contributions are

$$A_{ff}^{(1)}(\hat{u}_1, \hat{u}_2) = -\frac{1}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - 2 \right] \\ - \frac{1}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - 2 \right], \quad (\text{C.1})$$

$$A_{ff}^{(3)}(\hat{u}_1, \hat{u}_2) = \frac{1}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\ - \frac{1}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \quad (\text{C.2})$$

$$A_{ff}^{(4)}(\hat{u}_1, \hat{u}_2) = -\frac{1}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\ - \frac{1}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \quad (\text{C.3})$$

while the subleading coefficients read

$$B_{ff}^{(1)}(\hat{u}_1, \hat{u}_2) = -\frac{1}{(\hat{u}_1 - \hat{u}_2)^2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\ - \frac{1}{(\hat{u}_1 + \hat{u}_2)^2} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right]$$

$$C_{\text{ff}}^{(4)}(\hat{u}_1, \hat{u}_2) = \frac{1 + \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\ + \frac{1 - \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{\hat{u}_1 - 1}{\hat{u}_1 + 1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{\hat{u}_1 + 1}{\hat{u}_1 - 1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right]. \quad (\text{C.9})$$

C.2 Gluon-gluon case

For the gauge-gauge case, the $1/g$ contribution to phases are

$$A_{\text{GG}}^{(1)}(\hat{u}_1, \hat{u}_2) = \frac{2\mathcal{P}}{\hat{u}_1 - \hat{u}_2} + 2\pi i \delta(\hat{u}_1 + \hat{u}_2) \\ - \frac{\mathcal{P}}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} \right] \\ - \frac{\mathcal{P}}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} \right], \quad (\text{C.10})$$

$$A_{\text{GG}}^{(3)}(\hat{u}_1, \hat{u}_2) = -\frac{2i\mathcal{P}}{\hat{u}_1 + \hat{u}_2} - 2\pi \delta(\hat{u}_1 + \hat{u}_2) \\ + \frac{\mathcal{P}}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} \right] \\ - \frac{\mathcal{P}}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} \right], \quad (\text{C.11})$$

$$A_{\text{GG}}^{(4)}(\hat{u}_1, \hat{u}_2) = -\frac{2i\mathcal{P}}{\hat{u}_1 + \hat{u}_2} - 2\pi \delta(\hat{u}_1 + \hat{u}_2) \\ - \frac{\mathcal{P}}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} \right] \\ - \frac{\mathcal{P}}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} \right], \quad (\text{C.12})$$

and $1/g^2$ corrections take the form

$$B_{\text{GG}}^{(1)}(\hat{u}_1, \hat{u}_2) = \frac{\mathcal{P}}{[(\hat{u}_1 - \hat{u}_2)^2]_+} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} \right] \\ + \frac{\mathcal{P}}{[(\hat{u}_1 + \hat{u}_2)^2]_+} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} + 2i \right] \\ + \frac{2 - \hat{u}_1^2 - \hat{u}_2^2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \frac{\mathcal{P}}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} \right] \\ + \frac{2 - \hat{u}_1^2 - \hat{u}_2^2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \frac{\mathcal{P}}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{1 - \hat{u}_2}{1 + \hat{u}_2} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{1 + \hat{u}_2}{1 - \hat{u}_2} \right)^{1/4} \right]$$

We used above the Hadamard regularization which is also known (to physicists) as the so-called +-prescription. We verified that $f_{\text{GG}}^{(1)}(u_1, u_2) = f_{\text{GG}}^{(2)}(u_2, u_1)$. The above expressions coincide with the string calculation of Ref. [35].

C.3 Fermion-gluon case

Finally, we quote the gauge-fermion phases. These are

$$A_{\text{Gf}}^{(1)}(\hat{u}_1, \hat{u}_2) = -\frac{1}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \sqrt{2} \right] \\ - \frac{1}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \sqrt{2} \right], \quad (\text{C.19})$$

$$A_{\text{Gf}}^{(2)}(\hat{u}_1, \hat{u}_2) = \frac{1}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \sqrt{2} \right] \\ - \frac{1}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \sqrt{2} \right], \quad (\text{C.20})$$

$$A_{\text{Gf}}^{(3)}(\hat{u}_1, \hat{u}_2) = \frac{1}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + i\sqrt{2} \right] \\ - \frac{1}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + i\sqrt{2} \right], \quad (\text{C.21})$$

$$A_{\text{Gf}}^{(4)}(\hat{u}_1, \hat{u}_2) = -\frac{1}{\hat{u}_1 - \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + i\sqrt{2} \right] \\ - \frac{1}{\hat{u}_1 + \hat{u}_2} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + i\sqrt{2} \right], \quad (\text{C.22})$$

and

$$B_{\text{Gf}}^{(1)}(\hat{u}_1, \hat{u}_2) = \frac{\hat{u}_1 + \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\ + \frac{\hat{u}_1 - \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \\ B_{\text{Gf}}^{(2)}(\hat{u}_1, \hat{u}_2) = -\frac{\hat{u}_1 + \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\ + \frac{\hat{u}_1 - \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \\ B_{\text{Gf}}^{(3)}(\hat{u}_1, \hat{u}_2) = -\frac{\hat{u}_1 + \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right]$$

$$\begin{aligned}
& + \frac{\hat{u}_1 - \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \\
B_{\text{Gf}}^{(4)}(\hat{u}_1, \hat{u}_2) &= \frac{\hat{u}_1 + \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\
& + \frac{\hat{u}_1 - \hat{u}_2}{4(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \\
\end{aligned} \tag{C.23}$$

with

$$\begin{aligned}
C_{\text{Gf}}^{(1)}(\hat{u}_1, \hat{u}_2) &= \frac{1 + \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\
& + \frac{1 - \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \\
C_{\text{Gf}}^{(2)}(\hat{u}_1, \hat{u}_2) &= -\frac{1 + \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\
& + \frac{1 - \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} - \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \\
C_{\text{Gf}}^{(3)}(\hat{u}_1, \hat{u}_2) &= -\frac{1 + \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\
& + \frac{1 - \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right], \\
C_{\text{Gf}}^{(4)}(\hat{u}_1, \hat{u}_2) &= \frac{1 + \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} \right] \\
& + \frac{1 - \hat{u}_1 \hat{u}_2}{(1 - \hat{u}_1^2)(1 - \hat{u}_2^2)} \left[\left(\frac{1 - \hat{u}_1}{1 + \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 - 1}{\hat{u}_2 + 1} \right)^{1/4} + \left(\frac{1 + \hat{u}_1}{1 - \hat{u}_1} \right)^{1/4} \left(\frac{\hat{u}_2 + 1}{\hat{u}_2 - 1} \right)^{1/4} \right]. \\
\end{aligned} \tag{C.24}$$

References

- [1] B. Basso, A. Sever, P. Vieira, “Spacetime and flux tube S-matrices at finite coupling for N=4 supersymmetric Yang-Mills theory,” *Phys. Rev. Lett.* **111** (2013) 9, 091602 [arXiv:1303.1396 [hep-th]].
- [2] L.F. Alday, D. Gaiotto, J. Maldacena, A. Sever, P. Vieira, “An operator product expansion for polygonal null Wilson loops,” *JHEP* **1104** (2011) 088 [arXiv:1006.2788 [hep-th]].
- [3] L.F. Alday, J.M. Maldacena, “Gluon scattering amplitudes at strong coupling,” *JHEP* **0706** (2007) 064 [arXiv:0705.0303 [hep-th]].

- [4] J.M. Drummond, J. Henn, G.P. Korchemsky, E. Sokatchev, “On planar gluon amplitudes/Wilson loops duality,” Nucl. Phys. B **795** (2008) 52 [arXiv:0709.2368 [hep-th]].
- [5] A. Brandhuber, P. Heslop, G. Travaglini, “MHV amplitudes in N=4 super Yang-Mills and Wilson loops,” Nucl. Phys. B **794** (2008) 231 [arXiv:0707.1153 [hep-th]].
- [6] S. Caron-Huot, “Notes on the scattering amplitude/Wilson loop duality,” JHEP **1107** (2011) 058 [arXiv:1010.1167 [hep-th]].
- [7] L.J. Mason, D. Skinner, “The complete planar S-matrix of N=4 SYM as a Wilson loop in twistor space,” JHEP **1012** (2010) 018 [arXiv:1009.2225 [hep-th]].
- [8] A.V. Belitsky, G.P. Korchemsky, E. Sokatchev, “Are scattering amplitudes dual to super Wilson loops?,” Nucl. Phys. B **855** (2012) 333 [arXiv:1103.3008 [hep-th]].
- [9] B. Basso, “Exciting the GKP string at any coupling,” Nucl. Phys. B **857** (2012) 254 [arXiv:1010.5237 [hep-th]].
- [10] B. Basso, A. Sever, P. Vieira, “Space-time S-matrix and flux tube S-matrix II. Extracting and matching data,” JHEP **1401** (2014) 008 [arXiv:1306.2058 [hep-th]].
- [11] A.V. Belitsky, S.E. Derkachov, A.N. Manashov, “Quantum mechanics of null polygonal Wilson loops,” Nucl. Phys. B **882** (2014) 303 [arXiv:1401.7307 [hep-th]].
- [12] B. Basso, A. Sever, P. Vieira, “Space-time S-matrix and flux-tube S-matrix III. The two-particle contributions,” JHEP **1408** (2014) 085 [arXiv:1402.3307 [hep-th]].
- [13] A.V. Belitsky, “Nonsinglet pentagons and NMHV amplitudes,” Nucl. Phys. B **896** (2015) 493 [arXiv:1407.2853 [hep-th]].
- [14] B. Basso, A. Sever, P. Vieira, “Space-time S-matrix and flux-tube S-matrix IV. Gluons and fusion,” JHEP **1409** (2014) 149 [arXiv:1407.1736 [hep-th]].
- [15] A.V. Belitsky, “Fermionic pentagons and NMHV hexagon,” Nucl. Phys. B **894** (2015) 108 [arXiv:1410.2534 [hep-th]].
- [16] A.V. Belitsky, “On factorization of multiparticle pentagons,” Nucl. Phys. B **897** (2015) 346 [arXiv:1501.06860 [hep-th]].
- [17] B. Basso, J. Caetano, L. Cordova, A. Sever, P. Vieira, “OPE for all helicity amplitudes,” arXiv:1412.1132 [hep-th].
- [18] B. Basso, J. Caetano, L. Cordova, A. Sever, P. Vieira, “OPE for all helicity amplitudes II. Form factors and data analysis,” arXiv:1508.02987 [hep-th].
- [19] B. Basso, A. Sever, P. Vieira, “Collinear limit of scattering amplitudes at strong coupling,” Phys. Rev. Lett. **113** (2014) 26, 261604 [arXiv:1405.6350 [hep-th]].
- [20] D. Fioravanti, S. Piscaglia, M. Rossi, “Asymptotic Bethe Ansatz on the GKP vacuum as a defect spin chain: scattering, particles and minimal area Wilson loops,” Nucl. Phys. B **898** (2015) 301 [arXiv:1503.08795 [hep-th]].

- [21] L.J. Dixon, J.M. Drummond, J.M. Henn, “Analytic result for the two-loop six-point NMHV amplitude in $N=4$ super Yang-Mills theory,” JHEP **1201** (2012) 024 [arXiv:1111.1704 [hep-th]].
- [22] L.J. Dixon, M. von Hippel, “Bootstrapping an NMHV amplitude through three loops,” JHEP **1410** (2014) 65 [arXiv:1408.1505 [hep-th]].
- [23] J. Golden, M.F. Paulos, M. Spradlin, A. Volovich, “Cluster polylogarithms for scattering amplitudes,” J. Phys. A **47** (2014) 47, 474005 [arXiv:1401.6446 [hep-th]].
- [24] J. Golden, M. Spradlin, “An analytic result for the two-loop seven-point MHV amplitude in $\mathcal{N} = 4$ SYM,” JHEP **1408** (2014) 154 [arXiv:1406.2055 [hep-th]].
- [25] J. Golden, M. Spradlin, “A cluster bootstrap for two-loop MHV amplitudes,” JHEP **1502** (2015) 002 [arXiv:1411.3289 [hep-th]].
- [26] J.M. Drummond, G. Papathanasiou, M. Spradlin, “A symbol of uniqueness: the cluster bootstrap for the 3-loop MHV heptagon,” JHEP **1503** (2015) 072 [arXiv:1412.3763 [hep-th]].
- [27] L.F. Alday, D. Gaiotto, J. Maldacena, “Thermodynamic Bubble Ansatz,” JHEP **1109** (2011) 032 [arXiv:0911.4708 [hep-th]].
- [28] L.F. Alday, J. Maldacena, A. Sever, P. Vieira, “Y-system for scattering amplitudes,” J. Phys. A **43** (2010) 485401 [arXiv:1002.2459 [hep-th]].
- [29] Folklore.
- [30] B. Basso, G.P. Korchemsky, J. Kotanski, “Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling,” Phys. Rev. Lett. **100** (2008) 091601 [arXiv:0708.3933 [hep-th]].
- [31] B. Basso, G.P. Korchemsky, “Nonperturbative scales in AdS/CFT,” J. Phys. A **42** (2009) 254005 [arXiv:0901.4945 [hep-th]].
- [32] S. G. Mikhlin, “Linear integral equations”, Gordon & Breach (New York, 1961)
- [33] D. Fioravanti, S. Piscaglia, M. Rossi, “On the scattering over the GKP vacuum,” Phys. Lett. B **728** (2014) 288 [arXiv:1306.2292 [hep-th]].
- [34] L. Bianchi, M.S. Bianchi, “Worldsheet scattering for the GKP string,” JHEP **1511** (2015) 178 [arXiv:1508.07331 [hep-th]].
- [35] L. Bianchi, M.S. Bianchi, “On the scattering of gluons in the GKP string,” JHEP **1602** (2016) 146 [arXiv:1511.01091 [hep-th]].
- [36] B. Basso, A.V. Belitsky, “Luescher formula for GKP string,” Nucl. Phys. B **860** (2012) 1 [arXiv:1108.0999 [hep-th]].
- [37] B. Basso, A. Rej, “Bethe Ansatzes for GKP strings,” Nucl. Phys. B **879** (2014) 162 [arXiv:1306.1741 [hep-th]].

- [38] N. Beisert, B. Eden and M. Staudacher, “Transcendentality and crossing,” J. Stat. Mech. **0701** (2007) P01021 [hep-th/0610251].
- [39] L.F. Alday, G. Arutyunov, M.K. Benna, B. Eden, I.R. Klebanov, “On the strong coupling scaling dimension of high spin operators,” JHEP **0704** (2007) 082 [hep-th/0702028 [hep-th]].
- [40] A.V. Kotikov, L.N. Lipatov, “On the highest transcendentality in N=4 SUSY,” Nucl. Phys. B **769** (2007) 217 [hep-th/0611204].
- [41] I. Kostov, D. Serban, D. Volin, “Strong coupling limit of Bethe Ansatz equations,” Nucl. Phys. B **789** (2008) 413 [hep-th/0703031 [HEP-TH]].
- [42] M. Beccaria, G.F. De Angelis, V. Forini, “The scaling function at strong coupling from the quantum string Bethe equations,” JHEP **0704** (2007) 066 [hep-th/0703131].
- [43] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, “A semiclassical limit of the gauge/string correspondence,” Nucl. Phys. B **636** (2002) 99 [hep-th/0204051].
- [44] M. Kruczenski, “A note on twist two operators in N=4 SYM and Wilson loops in Minkowski signature,” JHEP **0212** (2002) 024 [hep-th/0210115].
- [45] S. Frolov, A.A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS(5) x S**5,” JHEP **0206** (2002) 007 [hep-th/0204226].
- [46] S. Frolov, A. Tirziu, A.A. Tseytlin, “Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT,” Nucl. Phys. B **766** (2007) 232 [hep-th/0611269].
- [47] R. Roiban, A. Tirziu, A.A. Tseytlin, “Two-loop world-sheet corrections in AdS(5) x S**5 superstring,” JHEP **0707** (2007) 056 [arXiv:0704.3638 [hep-th]];
R. Roiban, A.A. Tseytlin, “Strong-coupling expansion of cusp anomaly from quantum superstring,” JHEP **0711** (2007) 016 [arXiv:0709.0681 [hep-th]].
- [48] P.Y. Casteill, C. Kristjansen, “The strong coupling limit of the scaling function from the quantum string Bethe Ansatz,” Nucl. Phys. B **785** (2007) 1 [arXiv:0705.0890 [hep-th]].
- [49] A.V. Belitsky, “Strong coupling expansion of Baxter equation in N=4 SYM,” Phys. Lett. B **659** (2008) 732 [arXiv:0710.2294 [hep-th]].
- [50] S. Caron-Huot, S. He, “Jumpstarting the all-loop S-Matrix of planar N=4 super Yang-Mills,” JHEP **1207** (2012) 174 [arXiv:1112.1060 [hep-th]].
- [51] M. Bullimore, D. Skinner, “Descent Equations for superamplitudes,” arXiv:1112.1056 [hep-th].
- [52] A.V. Belitsky, “Descent equation for superloop and cyclicity of OPE,” arXiv:1506.02598 [hep-th].
- [53] B. Basso, A. Sever, P. Vieira, “Hexagonal Wilson loops in planar $\mathcal{N} = 4$ SYM theory at finite coupling,” arXiv:1508.03045 [hep-th].

- [54] A.V. Belitsky, “Nonperturbative enhancement of superloop at strong coupling,” arXiv:1512.00555 [hep-th].
- [55] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, “Integrals and series, Vol. 3: More Special functions”, Gordon & Breach (New York, 1989)